Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions

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Abstract: We prove the existence of Cantor families of periodic solutions for nonlinear wave equations in higher spatial dimensions with periodic boundary conditions. We study both forced and autonomous PDEs. In the latter case our theorems generalize previous results of Bourgain to more general nonlinearities of class C^k and assuming weaker non-resonance conditions. Our solutions have Sobolev regularity both in time and space. The proofs are based on a differentiable Nash-Moser iteration scheme, where it is sufficient to get estimates of interpolation type for the inverse linearized operators. Our approach works also in presence of very large "clusters of small divisors".

Keywords: Nonlinear Wave Equation, Infinite Dimensional Hamiltonian Systems, Nash-Moser Theorem, Periodic Solutions, Lyapunov-Schmidt reduction, small divisors[1].

2000AMS subject classification: 35L05, 37K50, 37K55.

1 Introduction

The search for periodic solutions of nonlinear wave equations has a long standing tradition. The first pioneering results of Rabinowitz [23] and Brezis-Coron-Nirenberg [5] proved, by means of global variational methods, the existence of periodic solutions for 1-dimensional nonlinear wave equations, with a rational frequency. The reason for such a condition is that the other frequencies give rise to a "small divisors" problem type, due to complex resonance phenomena.

On the other hand, the existence of periodic and quasi periodic solutions in a neighborhood of an elliptic equilibrium, for positive measure sets of frequencies, was also considered. In this direction, the first results have been proved by Kuksin [16] and Wayne [25] for one dimensional, analytic, nonlinear wave equations. The main difficulty, namely the presence of arbitrarily "small divisors" in the expansion series of the solutions, is handled via KAM theory. The pioneering results in [16]-[25] were limited to Dirichlet boundary conditions because they required the eigenvalues of the Laplacian to be *simple* (the square roots of the eigenvalues are the normal modes frequencies of small oscillations of the string). In this case one can impose strong non-resonance conditions between the "tangential" and the "normal" frequencies of the expected KAM torus (the so-called "second order Melnikov" non-resonance conditions) to solve the linear homological equations which arise at each step of the KAM iteration, see also [22]-[18]-[17]. Such equations are linear PDEs with constant coefficients and can be solved by standard Fourier series. For periodic boundary conditions, where two consecutive eigenvalues are possibly equal, the second order Melnikov non-resonance conditions are violated.

In order to overcome such limitations, Craig and Wayne [11] introduced the Lyapunov-Schmidt decomposition method for PDEs and solved the small divisors problem, for periodic solutions, with an analytic Newton iteration scheme. Such an approach is particularly designed for dealing with resonant situations. On the other hand, the main difficulty of this strategy lies in the inversion of the linearized operators obtained at each step of the iteration, and in achieving suitable estimates for their inverse in

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[1]Supported by MIUR Variational Methods and Nonlinear Differential Equations.

high (analytic) norms. Indeed these operators come from self-adjoint linear PDEs with *non-constant* coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. To solve this problem Craig and Wayne imposed, for positive measure sets of parameters, lower bounds for the moduli of the eigenvalues (in particular, all the eigenvalues must be non-zero). These assumptions imply upper bounds for the operatorial L^2-norm of the inverse operators. Next, in order to get estimates on the inverse in analytic norms, Craig and Wayne developed a coupling technique inspired by the methods of Frölich-Spencer [14] in the Anderson localization theory, see also [10]. Key ingredients to achieve estimates for the inverse operators in high norms are the following assumptions on the unperturbed operator: (i) "separation properties" between clusters of singular sites (that is between the Fourier indexes of the small divisors), (ii) properties of "well-localization" of the eigenfunctions with respect to the exponentials. These two facts, together with the analyticity of the functions, imply a very weak interaction between the singular sites. The second requirement (ii) is free when working with periodic boundary conditions. On the other hand, since the first requirement (i) can be obtained by imposing only the "first order Melnikov" non-resonance conditions, the Craig-Wayne approach works perfectly also in case of degenerate eigenvalues. In [11] the "clusters of small divisors" have a fixed bounded size: it is the case for one dimensional nonlinear wave and Schrödinger equations with periodic boundary conditions, see also [10].

The main difficulty in extending these results to PDEs in higher spatial dimensions is that the eigenvalues of the Laplacian can be highly degenerate, forming clusters of increasing size which tends to infinity.

This further problem has been first solved by Bourgain [6] for nonlinear wave equations in dimension $d \geq 2$ with periodic boundary conditions, extending the Craig-Wayne techniques. These results hold for analytic (polynomial) nonlinearities and prove the existence of periodic solutions having Gevrey regularity both in time and space. Suitable separations properties between the clusters of small divisors are imposed in [6] assuming a strong Diophantine-type condition, see remark 4.1. Then, using repeatedly the resolvent identity (see [7]), Bourgain proves a sub-exponentially fast decay for the off-diagonal terms of the inverse matrix. This estimate on the speed of decay gives an upper bound for the inverse matrix in Gevrey norm. For this step the high (Gevrey) regularity of the given functions is exploited. Similar results for the nonlinear Schrödinger equation have been proved in [7, Appendix 2], but, in that case, the separation conditions for the clusters of small divisors are more simply obtained.
The main result in [7] actually proves the existence also of quasi-periodic solutions in dimension $d = 2$. See also [8] about the construction of quasi-periodic orbits for the nonlinear wave and Schrödinger equations in any spatial dimension.

In the present paper we prove the existence of periodic solutions for higher dimensional nonlinear wave equations for merely differentiable nonlinearities and under weaker non-resonance conditions than in [6]. We consider both forced and autonomous PDEs. In the forced case Theorem 1.1 is the first higher dimensional result, and extends [21], [13], [2], which are valid for 1-dimensional equations. In the autonomous case Theorem 1.2 generalizes the result of Bourgain [6]. Our solutions have the same Sobolev regularity both in time and space.

In order to prove our results we need all the power of the differentiable Nash-Moser theory. In particular, the key point of the iterative process lies in the "a-priori" bounds (15) for the divergence of the high Sobolev norms of the approximate solutions; we refer to [4] for further explanations and for a comparison with the approach of [11].

Concerning the linearized operators obtained at each step of the Nash-Moser iteration, it is sufficient to achieve just *interpolation* type estimates for their inverses, see the key property (P5). Our approach works also in presence of possibly very large clusters of small divisors: the "dyadic" condition (H1) (see section 3.2) is weaker than the corresponding ones in [7] (Lemma 7 of Appendix 2) and in [15]. Furthermore (H1) could also be considerably weakened, see remark 3.1, even though not completely eliminated, see the discussion below and remark 3.2. A point of interest is that, in presence of possibly very large clusters of small divisors, the use of Sobolev norms, instead of analytic or Gevrey ones, used in [11]-[6], makes the estimates easier. The most intuitive reason is that a lower bound for the moduli of the eigenvalues yields immediately a L^2-bound for the inverse matrix, and the Sobolev norms are closer to the L^2-norm than the

Gevrey or analytic norms, see lemma 3.1 and related comments. Clearly, working with functions having a mere Sobolev regularity, has the drawback of a slow (polynomial) decay off the diagonal of the matrix elements of these operators. This makes the interactions between their singular clusters rather strong. Nevertheless a polynomial decay of large enough order (connected to some smoothness assumption) is sufficient.

On the other hand, we underline that it does not seem sufficient to have only lower bounds for the moduli of all the eigenvalues without some separation properties between the singular clusters. This information, by itself, would give a too weak estimate for the norm of the inverse matrix, and the Nash-Moser scheme would not converge, see remark 3.2. This is also related to a famous counter-example of Lojasiewicz-Zehnder [19] concerning the optimal conditions in an abstract Nash-Moser implicit function theorem.

Since the aim of the present paper is to focus on the solution of the small divisors problem in presence of large clusters and with differentiable nonlinearities, we have considered model cases in which the bifurcation equation arising with the Lyapunov-Schmidt reduction, or is not present (as for Theorem 1.1) or it is rather simply solved (as for Theorem 1.2).

Before concluding this introduction, we mention that the KAM approach has been extended by Chierchia-You [9] to prove the existence of quasi-periodic solutions for one dimensional wave and Schrödinger equations in case of periodic boundary conditions (in particular, their theorem covers the result of [11]), and by Eliasson-Kuksin [12] for higher dimensional nonlinear Schrödinger equations. We also mention that Gentile-Procesi [15] have recently obtained the existence of periodic solutions for higher dimensional nonlinear Schrödinger equations by the Lindstedt series method. We remark that in all the previous results the nonlinearities are required to be analytic and the solutions are analytic in time.

Acknowledgments: The authors thank P. Baldi, L. Biasco and M.Procesi for interesting comments.

1.1 Main results

Let us consider d-dimensional nonlinear wave equations with periodic boundary conditions of the form

$$\begin{cases} u_{tt} - \Delta u + mu = \varepsilon F(\omega t, x, u) \\ u(t,x) = u(t, x + 2\pi k)\,, \qquad \forall k \in \mathbf{Z}^d \end{cases} \tag{1}$$

where the forcing term $F(\omega t, x, u)$ is $2\pi/\omega$-periodic in time[2], 2π-periodic in each spatial variable x_i, $i = 1, \ldots, d$, $m \in \mathbf{R}$ and $\varepsilon > 0$ is a small parameter.

We consider the *non-resonant* case when

$$|\omega^2 l^2 - \lambda_j| \geq \frac{3\gamma}{\max(1, |l|^{3/2})}\,, \quad \gamma \in (0,1)\,, \quad \lambda_j := |j|^2 + m\,, \quad \forall\ (l,j) \in \mathbf{Z} \times \mathbf{Z}^d\,. \tag{2}$$

Note that in (2), the exponent $3/2$ is fixed for simplicity of exposition and could be replaced by any $\alpha > 1$. For all j such that $\lambda_j \geq 0$ define $\omega_j := \sqrt{\lambda_j}$. Assumption (2) means that the forcing frequency ω does not enter in resonance with the normal mode frequencies ω_j of oscillations of the membrane.

By standard arguments, (2) is satisfied for all ω in $[\bar\omega_1, \bar\omega_2]$ but a subset of measure $O(\gamma)$.

Concerning regularity we only assume that $F \in C^k(\mathbf{T} \times \mathbf{T}^d \times \mathbf{R}; \mathbf{R})$ for some k large enough.
If $F(t, x, 0) \not\equiv 0$ then $u = 0$ is not a solution of (1) for $\varepsilon \neq 0$.

- Question: do there exist periodic solutions of (1) for positive measure sets of (ε, ω)?

Normalizing the period, we look for 2π-periodic in time solutions of

$$\omega^2 u_{tt} - \Delta u + mu = \varepsilon F(t, x, u) \tag{3}$$

[2] That is $F(\cdot\,, x, u)$ is 2π-periodic.

(where F is 2π-periodic in t) in the real Sobolev space[3]

$$\begin{aligned} H^s := H^s(\mathbf{T}\times\mathbf{T}^d;\mathbf{R}) \quad := \quad & \Big\{ u(t,x) := \sum_{(l,j)\in\mathbf{Z}\times\mathbf{Z}^d} u_{l,j} e^{\mathrm{i}(lt+j\cdot x)}\,,\ u^*_{l,j} = u_{-l,-j} \\ & \mid\ \|u\|_s^2 := \sum_{(l,j)\in\mathbf{Z}\times\mathbf{Z}^d} |u_{l,j}|^2 \max((l^2+|j|^2)^s,1) < +\infty \Big\} \end{aligned}$$

for some $s > (d+1)/2$.

We recall that for $s > (d+1)/2$ we have the continuous inclusion $H^s(\mathbf{T}^{d+1}) \hookrightarrow L^\infty(\mathbf{T}^{d+1})$ and H^s is a Banach algebra with respect to the multiplication of functions.

Theorem 1.1 *Fix $0 < \bar\omega_1 < \bar\omega_2$. There is $s := s(d)$, $k := k(d) \in \mathbf{N}$, such that: $\forall F \in C^k(\mathbf{T}\times\mathbf{T}^d\times\mathbf{R})$, $\forall\gamma\in(0,1)$, there exist $\varepsilon_0 := \varepsilon_0(\gamma)$, $K, C > 0$ (independent of γ), a map*

$$\widetilde{u} \in C^1([0,\varepsilon_0]\times[\bar\omega_1,\bar\omega_2];H^s) \quad \text{with} \quad \|\widetilde{u}(\varepsilon,\omega)\|_s \le K\gamma^{-1}\varepsilon\,,\ \|(D_{\varepsilon,\omega}\widetilde{u})(\varepsilon,\omega)\|_s \le K\gamma^{-1}\,,$$

and a Cantor like set $A_\infty \subset [0,\varepsilon_0]\times[\bar\omega_1,\bar\omega_2]$, of Lebesgue measure

$$|A_\infty| \ \ge\ \varepsilon_0\,(|\bar\omega_2-\bar\omega_1| - C\gamma)\,, \tag{4}$$

such that, $\forall(\varepsilon,\omega)\in A_\infty$, $\widetilde{u}(\varepsilon,\omega)$ is a solution of (3).

Note that, as the freely chosen small parameter γ tends to 0, the constant $\varepsilon_0(\gamma)$ tends to 0, and the bounds on $\widetilde{u}$ get worse, but, as a counterpart, the ratio $|A_\infty|/\varepsilon_0(\bar\omega_2-\bar\omega_1)$ tends to 1, *i.e.* the set A_∞ has asymptotically full measure.

The conditions defining A_∞ are (2) and (7) (which are independent of ε), plus infinitely many others, which depend on the nonlinearity, and are required to get the invertibility of the linearized operators obtained at each step of the Nash-Moser iteration (see Theorem 2.1).

Remark 1.1 *The non-resonance condition (2) implies $m \ne 0$. Note that if $m < 0$ the equilibrium $u = 0$ is not completely elliptic. In the case $m = 0$, under some additional assumptions on F, a result similar to Theorem 1.1 holds assuming condition (2) only for all $(l,j)\ne 0$. Then we perform a Lyapunov-Schmidt reduction according to the decomposition $H^s = \mathbf{R}\oplus H^s_0$ where H^s_0 denote the Sobolev functions with zero mean value.*

We develop in detail all the computations to prove Theorem 1.1. The same techniques can be used to prove the existence of Cantor families of small amplitude periodic solutions for autonomous d-dimensional nonlinear wave equations of the form

$$\begin{cases} u_{tt} - \Delta u + mu = au^p + r(x,u) \\ u(t,x) = u(t,x+2\pi k)\,, \qquad \forall k \in \mathbf{Z}^d \end{cases} \tag{5}$$

with $p \ge 3$ odd integer, $a \in \mathbf{R}$, $a \ne 0$, and

$$r(x,u) \in C^k(\mathbf{T}^d\times\mathbf{R})\,,\ k \ge p\,, \quad r(x,0) = \ldots = \partial_u^p r(x,0) = 0 \quad \text{and} \quad r(-x,u) = r(x,u)\,. \tag{6}$$

The bounded solutions of the linearized equation $u_{tt} - \Delta u + mu = 0$ which are even in time and in x are

$$u = \sum_{j\in\mathbf{Z}^d,|j|^2+m\ge 0} \cos(\omega_j t) A_j \cos(j\cdot x)\,, \quad A_j \in \mathbf{R}\,, \quad \omega_j := \sqrt{|j|^2+m}\,.$$

Fixed j_0, we aim to prove the existence of small amplitude periodic solutions of the nonlinear equation (5) with frequencies close to ω_{j_0}. Assuming that m is irrational, the normal mode frequencies ω_j commensurable with ω_{j_0} satisfy $|j| = |j_0|$. We shall assume the stronger hypothesis that m is Diophantine, to have a quantitative non-resonance condition like

$$|\omega_j^2 - l^2\omega_{j_0}^2| \ge \frac{\gamma}{(|l|+1)^\tau}\,, \qquad \forall\,(|j|,|l|) \ne (|j_0|,1)\,,$$

[3]The symbol z^* denotes the complex conjugate of $z \in \mathbf{C}$.

similar to (2), see [6].

Rescaling the amplitude $u \mapsto \delta u$, $\delta > 0$ and normalizing the period, we look for solutions of

$$\omega^2 u_{tt} - \Delta u + mu = \varepsilon g(\delta, x, u)\,, \qquad \varepsilon := \delta^{p-1}\,, \qquad g(\delta, x, u) := au^p + \delta^{-p} r(x, \delta u)$$

in the subspace $H^s_{\rm even} := \{u \in H^s \mid u(-t,-x) = u(t,x)\}$. The regularity property of the composition operator induced by g on $H^s_{\rm even}$ are proved as in [4].

Theorem 1.2 *Let $p \geq 3$ be an odd integer and assume (6). Suppose $m > 0$ is Diophantine. Fix $j_0 \in \mathbf{Z}^d$. There is $s := s(d)$, $k := k(d) \in \mathbf{N}$ such that, for all $r \in C^k$, $\forall \gamma \in (0,1)$, there exist $\delta_0 > 0$, $\bar{A}_{j_0} \in \mathbf{R}$, a curve*

$$u \in C^1([0,\delta_0]; H^s_{\rm even}) \quad \text{with} \quad \|u(\delta) - \delta \bar{A}_{j_0} \cos(t)\cos(j_0 \cdot x)\|_s = O(\delta^2)\,,$$

and a Cantor set $\mathcal{C} \subset [0,\delta_0]$ of asymptotically full measure such that, $\forall \delta \in \mathcal{C}$, $u(\delta)$ is a solution of $\omega^2 u_{tt} - \Delta u + mu = u^p + r(x,u)$ with $\omega^2 = \omega_{j_0}^2 - \mathrm{sign}(a)\,\delta^{p-1}$.

Theorem 1.2 generalizes Bourgain's result in [6] to the case of differentiable nonlinearities, when the leading nonlinear term is au^p, $p \geq 3$ and for irrational frequencies satisfying only (7). Furthermore the existence result of theorem 1.2 holds for a Cantor set $\mathcal{C}$ of asymptotically full measure. This requires to write the dependence with respect to γ in the separation arguments of section 4 (used also in Theorem 1.1).

As said in the introduction the most difficult step in the proof of both Theorems 1.1-1.2 is to get estimates for the inverse linearized operators which arise at each step of the Nash-Moser iteration. For this task some "separation properties" for the "singular sites" seem required. For one dimensional wave equations, by a simple argument in [11]-[10], it is sufficient to assume that ω^2 is Diophantine, namely

$$|\omega^2 q - p| \geq \frac{\gamma}{\max(1, |p|^{3/2})}\,, \qquad \forall (q,p) \in \mathbf{Z}^2 \setminus \{(0,0)\} \tag{7}$$

with $\gamma \in (0,1)$. On the other hand, such separation properties are far from obvious in higher spatial dimension. They have been obtained by Bourgain in [6] under strong non-resonance conditions of Diophantine type for ω^2, see remark 4.1. In section 4 we shall obtain the same separation properties assuming only (7). Also in (7), the exponent $3/2$ is fixed only for simplicity and could be replaced by any $\alpha > 1$.

A final comment regarding the boundary conditions. The case of Dirichlet boundary conditions on a rectangle (with some oddness assumption on the nonlinearity) works similarly to the setting of the present case. The eigenfunctions of the unperturbed operator are still linear combinations of exponentials and the high order regularity of u has a straightforward translation into the behavior of its Fourier coefficients, *i.e.* of its components in the orthonormal basis of the eigenfunctions. On the contrary, in the case of a general bounded domain $\Omega \subset \mathbf{R}^d$, the eigenfunctions of the Laplacian do not possess such a good property, even if they form an orthonormal basis of $H^1(\Omega)$. Therefore the existence of periodic solutions is this case is completely open (but see [1] in the case of an integral nonlinearity which does not mix the spatial modes).

Notations: $\mathcal{N}(A,\eta)$ denotes the η-neighborhood of a subset A of a normed space; z^* is the complex conjugate of $z \in \mathbf{C}$; the symbol $[x] \in \mathbf{N}$ denotes the integer part of $x \in \mathbf{R}$. We denote by $\mathcal{L}(H_A, H_B)$ the set of continuous linear operators from H_A to H_B. $\mathrm{d}(A;B) := \inf\{|a-b|\,, a \in A,\, b \in B\}$ is the distance from the set A to the set B.

2 The Nash-Moser scheme

Consider the orthogonal splitting

$$H^s = W^{(N_n)} \oplus W^{(N_n)\perp}$$

where

$$W^{(N_n)} = \Big\{ u \in H^s \;\Big|\; u = \sum_{|(l,j)| \leq N_n} u_{l,j}\, e^{\mathrm{i}(lt + j\cdot x)} \Big\}$$

$$W^{(N_n)\perp} = \Big\{ u \in H^s \ \Big| \ u = \sum_{|(l,j)|>N_n} u_{l,j}\, e^{\mathrm{i}(lt+j\cdot x)} \Big\}$$

with[4]

$$N_n := [e^{\lambda 2^n}]\,, \qquad \lambda := \ln N_0\,. \tag{8}$$

In the proof of Theorem 2.1 we shall take λ, i.e. $N_0 \in \mathbf{N}$, large enough. We denote by

$$P_{N_n} : H^s \to W^{(N_n)} \qquad \text{and} \qquad P^{\perp}_{N_n} : H^s \to W^{(N_n)\perp}$$

the orthogonal projectors onto $W^{(N_n)}$ and $W^{(N_n)\perp}$.

The convergence of the Nash-Moser scheme is based on properties $(P1)$, $(P2)$, $(P3)$, $(P4)$, $(P5)$ below. The first three properties are standard for the composition operator $f : H^s \to H^s$ defined by

$$f(u)(t,x) := F(t,x,u(t,x))$$

where $F \in C^k(\mathbf{T} \times \mathbf{T}^d \times \mathbf{R}; \mathbf{R})$ with $k \geq s+2$ and $s > (d+1)/2$.

- **(P1)** (**Regularity**) $f \in C^2(H^s; H^s)$ and $D^2 f$ is bounded on $\{\|u\|_s \leq 1\}$.
- **(P2)** (**Tame**) $\forall\, s \leq s' \leq k$, $\forall u \in H^{s'}$ such that $\|u\|_s \leq 1$, $\|f(u)\|_{s'} \leq C(s')(1 + \|u\|_{s'})$.
- **(P3)** (**Taylor Tame**) $\forall s \leq s' \leq k-2$, $\forall u \in H^{s'}$ such that $\|u\|_s \leq 1$, $\forall h \in H^{s'}$,

$$\|f(u+h) - f(u) - Df(u)\,h\|_{s'} \leq C(s')(\|u\|_{s'}\|h\|_s^2 + \|h\|_s\|h\|_{s'})$$

where $[Df(u)h](t,x) = (\partial_u F)(t,x,u)h(t,x)$. In particular, for $s' = s$,

$$\|f(u+h) - f(u) - Df(u)\,h\|_s \leq C\|h\|_s^2\,. \tag{9}$$

We refer to [20] for the proof of $(P2)$, see also [24]. Properties $(P1)$ and $(P3)$ are obtained similarly. Furthermore, by the definitions of the spaces H^s and the projectors P_N, we have:

- **(P4)** (**Smoothing**) $\forall N \in \mathbf{N}\backslash\{0\}$, $\quad \begin{array}{ll} \|P_N u\|_{s+r} \leq N^r \|u\|_s\,, & \forall u \in H^s \\ \|P_N^{\perp} u\|_s \leq N^{-r}\|u\|_{s+r}\,, & \forall u \in H^{s+r}\,. \end{array}$

The key property $(P5)$, proved in section 3, is an *invertibility property* for the linearized operator

$$\mathcal{L}_N(\varepsilon,\omega,u(\varepsilon,\omega))[h] := L_\omega h - \varepsilon P_N Df(u(\varepsilon,\omega))\,h\,, \qquad \forall h \in W^{(N)} \tag{10}$$

where

$$L_\omega := \omega^2 \partial_{tt} - \Delta + m$$

and $u \in C^1([0,\varepsilon_0] \times [\bar\omega_1, \bar\omega_2], W^{(N)})$ with $\|u\|_{C^1(s)} := \sup_{[0,\varepsilon_0]\times[\bar\omega_1,\bar\omega_2]} \|u(\varepsilon,\omega)\|_s + \|D_{\varepsilon,\omega} u(\varepsilon,\omega)\|_s$.

Let

$$G := \Big\{ (\varepsilon,\omega) \in [0,\varepsilon_0] \times [\bar\omega_1,\bar\omega_2] \ \Big| \ \omega \text{ satisfies (2) and (7)} \Big\}\,.$$

- **(P5) (Invertibility of $\mathcal{L}_N$)** $\exists\, \mu := \mu(d)$, $s := s(d)$, $\bar C$, c, K, such that $\forall \gamma \in (0,1)$, $\forall s' \geq s$, $\forall C > 0$, there exist $\varepsilon_0 := \varepsilon_0(\gamma, s', C) > 0$ with the following property:
  $\forall N$, $\forall u \in C^1([0,\varepsilon_0] \times [\bar\omega_1,\bar\omega_2], W^{(N)})$ with $\|u\|_{C^1(s)} \leq C\gamma^{-1}$, there is a set $G_N(u) \subset [0,\varepsilon_0] \times [\bar\omega_1,\bar\omega_2]$ such that, if $(\varepsilon,\omega) \in \mathcal{N}(G_N(u), \gamma N^{-\sigma})$, $\sigma := \mu + 3$, then $\mathcal{L}_N(\varepsilon,\omega,u(\varepsilon,\omega))$ is invertible and

$$\Big\|\mathcal{L}_N^{-1}(\varepsilon,\omega,u)[h]\Big\|_{s'} \leq \frac{K(s')}{\gamma} N^\mu \Big(\|h\|_{s'} + \|u\|_{s'}\|h\|_s \Big)\,, \qquad \forall h \in W^{(N)}\,, \tag{11}$$

$$\Big\|\mathcal{L}_N^{-1}(\varepsilon,\omega,u)[h]\Big\|_s \leq \frac{K}{\gamma} N^\mu \|h\|_s\,. \tag{12}$$

Moreover, if $N \leq c(\gamma\varepsilon_0^{-1})^{2/3}$ then $G_N(u) = G$, and if $\|u_1 - u_2\|_s \leq N^{-\sigma}$, then

$$|G^c_{N'}(u_2) \backslash G^c_N(u_1)| \leq \bar C \frac{\gamma \varepsilon_0}{N}\,, \qquad \forall N' \geq N\,. \tag{13}$$

[4]The symbol $[\,\cdot\,]$ denotes the integer part.

Theorem 2.1 (Nash-Moser) *Let $0 < \varepsilon \le \varepsilon_0(\gamma)$ be small enough (possibly depending on d, F, $\bar{\omega}_1$, $\bar{\omega}_2$). There exists a decreasing sequence of subsets of "non-resonant" parameters*

$$\ldots \subseteq A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0 := G \subset A := [0, \varepsilon_0] \times [\bar{\omega}_1, \bar{\omega}_2]$$

and a sequence of functions $\widetilde{u}_n(\varepsilon, \omega) \in C^1(A, W^{(N_n)})$ satisfying $\|\widetilde{u}_n\|_s \le K\varepsilon\gamma^{-1}$, $\|D_{\varepsilon,\omega}\widetilde{u}_n\|_s \le K\gamma^{-1}$, such that, $\forall n$, if $(\varepsilon, \omega) \in \mathcal{N}(A_n, \gamma N_n^{-\sigma}/2)$ then $\widetilde{u}_n(\varepsilon, \omega)$ is a solution of

$$L_\omega u - \varepsilon P_{N_n} f(u) = 0 \,. \tag{P_{N_n}}$$

On the Cantor like set $A_\infty := \cap_{n\ge 0} A_n$, which satisfies the measure estimate (4), the sequence $(\widetilde{u}_n(\varepsilon, \omega))$ converges in H^s to a solution $\widetilde{u}(\varepsilon, \omega)$ of equation (3) satisfying $\|\widetilde{u}\|_s \le K\varepsilon\gamma^{-1}$, $\|D_{\varepsilon,\omega}\widetilde{u}\|_s \le K\gamma^{-1}$. The convergence is uniform in $(\varepsilon, \omega) \in A_\infty$.

Proof. In the proof K, K', K_i, denote constants depending on d, F, N_0, $\bar{\omega}_1$, $\bar{\omega}_2$ at most.

First step: initialization. Assume $\sigma > 7/2$. If $(\varepsilon, \omega) \in \mathcal{N}(G, \gamma N_0^{-\sigma})$ then, by (2), $\forall\, |(l, j)| \le N_0$, $|\omega^2 l^2 - \lambda_j| \ge \gamma/N_0^{3/2}$, for N_0 large so that $N_0^{\sigma-(7/2)} > \bar{\omega}_2$. Hence $\|L_\omega^{-1} h\|_s \le N_0^{3/2}\gamma^{-1}\|h\|_s$, $\forall h \in W^{(N_0)}$.

By the contraction mapping theorem, using $(P1)$, for $\varepsilon\gamma^{-1}$ small, there exists a unique solution $u_0 := u_0(\varepsilon, \omega)$ of equation (P_{N_0}) satisfying $\|u_0\|_s \le K_0\varepsilon\gamma^{-1}$. Furthermore, by the implicit function theorem, $(\varepsilon, \omega) \mapsto u_0(\varepsilon, \omega)$ is in $C^1(\mathcal{N}(G, \gamma N_0^{-\sigma}), H^s)$ and $\|D_{\varepsilon,\omega} u_0\|_s \le K_0\gamma^{-1}$.

We define $\widetilde{u}_0 := \psi_0 u_0$ where ψ_0 is a C^∞ cut-off function defined on A that takes the values 1 on $\mathcal{N}(G, \gamma N_0^{-\sigma}/2)$ and 0 outside $\mathcal{N}(G, \gamma N_0^{-\sigma})$, and $|D\psi_0| \le C N_0^\sigma \gamma^{-1}$. By the previous estimates, $\|\widetilde{u}_0\|_s \le K\varepsilon\gamma^{-1}$, $\|D_{\varepsilon,\omega}\widetilde{u}_0\|_s \le K\gamma^{-1}$, taking $\varepsilon C N_0^\sigma \gamma^{-1} < 1$.

Second step: iteration. Suppose we have already defined $\widetilde{u}_n \in C^1(A, W^{(N_n)})$ satisfying the properties stated in the proposition and, in addition, $\forall 1 \le k \le n$,

$$\|\widetilde{u}_k - \widetilde{u}_{k-1}\|_s \le K\varepsilon\gamma^{-1} N_k^{-\sigma-1}\,, \qquad \|D_{\varepsilon,\omega}\widetilde{u}_k - D_{\varepsilon,\omega}\widetilde{u}_{k-1}\|_s \le K\gamma^{-1} N_k^{-1} \tag{14}$$

$$B_k \le (1 + N_k^\mu) B_{k-1}\,, \qquad B'_k \le B'_{k-1} + K\gamma^{-1} N_k^\mu (N_k^\sigma B_k + \varepsilon B'_{k-1}) \tag{15}$$

where

$$B_n := 1 + \|\widetilde{u}_n\|_{s+\beta}\,, \qquad B'_n := 1 + \|D_{\varepsilon,\omega}\widetilde{u}_n\|_{s+\beta}$$

and

$$\beta := 2(\sigma + 1 + 3\mu)\;, \qquad \sigma = \mu + 3\,. \tag{16}$$

By the first inequality in (15) we get

$$B_n \le B_0 \Pi_{i=1}^n (1 + N_i^\mu) \le K N_{n+1}^\mu \tag{17}$$

and so, by the second inequality,

$$B'_n \le K\gamma^{-1} N_{n+1}^{2\mu+\sigma}\,. \tag{18}$$

For $h \in W^{(N_{n+1})}$ we write

$$\begin{aligned} L_\omega(\widetilde{u}_n + h) - \varepsilon P_{N_{n+1}} f(\widetilde{u}_n + h) &= \Big[L_\omega \widetilde{u}_n - \varepsilon P_{N_{n+1}} f(\widetilde{u}_n)\Big] + \Big[L_\omega h - \varepsilon P_{N_{n+1}} Df(\widetilde{u}_n) h\Big] + R_n(h) \\ &= r_n + \mathcal{L}_{N_{n+1}}(\varepsilon, \omega, \widetilde{u}_n) h + R_n(h) \end{aligned}$$

where

$$r_n := L_\omega \widetilde{u}_n - \varepsilon P_{N_{n+1}} f(\widetilde{u}_n) \qquad \text{and} \qquad R_n(h) := -\varepsilon P_{N_{n+1}}(f(\widetilde{u}_n + h) - f(\widetilde{u}_n) - Df(\widetilde{u}_n) h)\,.$$

If $(\varepsilon, \omega) \in \mathcal{N}(A_n; \gamma N_n^{-\sigma}/2)$ then $\widetilde{u}_n$ solves equation (P_{N_n}) and so

$$r_n := L_\omega \widetilde{u}_n - \varepsilon P_{N_{n+1}} f(\widetilde{u}_n) = -\varepsilon P_{N_n}^\perp P_{N_{n+1}} f(\widetilde{u}_n) \in W^{(N_n)\perp} \cap W^{(N_{n+1})}\,. \tag{19}$$

Inversion of $\mathcal{L}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)$. By property (P5), defining

$$A_{n+1} := A_n \cap G_{N_{n+1}}(\widetilde{u}_n)\,, \tag{20}$$

$\mathcal{L}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)$ is invertible for $(\varepsilon,\omega)\in\mathcal{N}(A_{n+1},\gamma N_{n+1}^{-\sigma})$ and

$$\Big\|\mathcal{L}^{-1}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)h\Big\|_s \overset{(12)}{\leq} \frac{K}{\gamma}N^{\mu}_{n+1}\|h\|_s\,, \qquad \forall h\in W^{(N_{n+1})}\,. \tag{21}$$

Set

$$\mathcal{G}_{n+1}: W^{(N_{n+1})}\to W^{(N_{n+1})}\,, \qquad \mathcal{G}_{n+1}(h) := -\mathcal{L}^{-1}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)(r_n+R_n(h))$$

so that equation $(P_{N_{n+1}})$ is equivalent to the the fixed point problem $h=\mathcal{G}_{n+1}(h)$.

Lemma 2.1 (Contraction) $\exists K_1>0$ *such that, for* $\varepsilon\gamma^{-1}$ *small enough,* $\forall(\varepsilon,\omega)\in\mathcal{N}(A_{n+1},\gamma N^{-\sigma}_{n+1})$, $\mathcal{G}_{n+1}$ *is a contraction in* $\mathcal{B}_{n+1} := \{h\in W^{(N_{n+1})}\mid \|h\|_s\leq\rho_{n+1} := 2K_1\varepsilon\gamma^{-1}N^{-\sigma-1}_{n+1}\}$.

PROOF. For N_0 large enough (i.e. λ large enough, see (8)) we have $N^{-\sigma}_{n+1} < N_n^{-\sigma}/2$, $\forall n$, and so $\mathcal{N}(A_{n+1},\gamma N^{-\sigma}_{n+1})\subset\mathcal{N}(A_n;\gamma N_n^{-\sigma}/2)$. Then (19) holds and

$$\begin{aligned}
\Big\|\mathcal{G}_{n+1}(h)\Big\|_s &\overset{(21)}{\leq} \frac{K}{\gamma}N^{\mu}_{n+1}\Big(\|r_n\|_s+\|R_n(h)\|_s\Big)\\
&\overset{(P4),(9)}{\leq} \frac{K'}{\gamma}N^{\mu}_{n+1}\Big(\varepsilon\,N_n^{-\beta}\|P_{N_{n+1}}f(\widetilde{u}_n)\|_{s+\beta}+\varepsilon\,\|h\|_s^2\Big)\\
&\overset{(P2)}{\leq} \frac{\varepsilon}{\gamma}K"N^{\mu}_{n+1}\Big(N_n^{-\beta}B_n+\|h\|_s^2\Big)\\
&\overset{(17)}{\leq} \frac{\varepsilon}{\gamma}K_1N^{\mu}_{n+1}N_n^{-\beta}N^{\mu}_{n+1}+\frac{\varepsilon}{\gamma}K_1N^{\mu}_{n+1}\|h\|_s^2\,.
\end{aligned} \tag{22}$$

By (22) and the definition of β in (16), if $\|h\|_s\leq\rho_{n+1} := 2K_1\varepsilon\gamma^{-1}N^{-\sigma-1}_{n+1}$, then

$$\Big\|\mathcal{G}_{n+1}(h)\Big\|_s \leq \frac{\varepsilon}{\gamma}K_1N^{-\sigma-1}_{n+1}+\frac{\varepsilon}{\gamma}K_1\,N^{\mu}_{n+1}\,\rho^2_{n+1}\leq\rho_{n+1}$$

for $\varepsilon\gamma^{-1}$ small enough.

Next we use the formula

$$D\mathcal{G}_{n+1}(h)[v] = \varepsilon\mathcal{L}^{-1}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)P_{N_{n+1}}\Big((Df(\widetilde{u}_n+h)-Df(\widetilde{u}_n))v\Big)$$

to obtain

$$\|D\mathcal{G}_{n+1}(h)[v]\|_s \leq K\frac{\varepsilon}{\gamma}N^{\mu}_{n+1}\|h\|_s\|v\|_s \leq K\frac{\varepsilon}{\gamma}N^{\mu}_{n+1}\rho_{n+1}\|v\|_s \leq \frac{\|v\|_s}{2} \tag{23}$$

for $\varepsilon\gamma^{-1}$ small enough. Hence $\mathcal{G}_{n+1}$ is a contraction on $\mathcal{B}_{n+1}$. ■

Let h_{n+1} be the fixed point of $\mathcal{G}_{n+1}$ defined for all $(\varepsilon,\omega)\in\mathcal{N}(A_{n+1},\gamma N^{-\sigma}_{n+1})$ and $u_{n+1} := \widetilde{u}_n+h_{n+1}$.

Lemma 2.2 (Estimate of the derivatives) *The map* $h_{n+1}\in C^1(\mathcal{N}(A_{n+1},\gamma N^{-\sigma}_{n+1});W^{(N_{n+1})})$ *and* $\|D_{\varepsilon,\omega}h_{n+1}\|_s\leq K\gamma^{-1}N^{-1}_{n+1}$.

PROOF. For all $(\varepsilon,\omega)\in\mathcal{N}(A_{n+1},\gamma N^{-\sigma}_{n+1})$, $h_{n+1}(\varepsilon,\omega)$ is a solution of $U_{n+1}(\varepsilon,\omega,h_{n+1}(\varepsilon,\omega))=0$ where

$$U_{n+1}(\varepsilon,\omega,h) := L_\omega(\widetilde{u}_n+h)-\varepsilon P_{N_{n+1}}f(\widetilde{u}_n+h)\,.$$

Note that, with the notations of the proof of lemma 2.1, $U_{n+1}(\varepsilon,\omega,h)=\mathcal{L}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)(h-\mathcal{G}_{n+1}(h))$. Since $\|D\mathcal{G}_{n+1}(h_{n+1})\|_s\leq 1/2$ (see (23)), $D_hU_{n+1}(\varepsilon,\omega,h_{n+1})=\mathcal{L}_{N_{n+1}}(\varepsilon,\omega,u_{n+1})$ is invertible and

$$\Big\|\Big(D_hU_{n+1}(\varepsilon,\omega,h_{n+1})\Big)^{-1}\Big\|_s = \Big\|(I-D\mathcal{G}_{n+1}(h_{n+1}))^{-1}\mathcal{L}^{-1}_{N_{n+1}}(\varepsilon,\omega,\widetilde{u}_n)\Big\|_s \leq \frac{K}{\gamma}N^{\mu}_{n+1}\,. \tag{24}$$

Hence, by the Implicit Function Theorem, $h_{n+1} \in C^1(\mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma}), W^{(N_{n+1})})$ and

$$D_{\varepsilon,\omega} h_{n+1} = -\mathcal{L}_{N_{n+1}}^{-1}(\varepsilon, \omega, u_{n+1})(D_{\varepsilon,\omega} U_{n+1})(\varepsilon, \omega, h_{n+1}) . \tag{25}$$

Now, using that $\widetilde{u}_n(\varepsilon, \omega)$ solves (P_{N_n}) for $(\varepsilon, \omega) \in \mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma})$, we get

$$D_\varepsilon U_{n+1}(\varepsilon, \omega, h_{n+1}) = P_{N_n} f(\widetilde{u}_n) - P_{N_{n+1}} f(u_{n+1}) + \varepsilon P_{N_n} Df(\widetilde{u}_n) D_\varepsilon \widetilde{u}_n - \varepsilon P_{N_{n+1}} Df(u_{n+1}) D_\varepsilon \widetilde{u}_n \tag{26}$$

and

$$D_\omega U_{n+1}(\varepsilon, \omega, h_{n+1}) = 2\omega (h_{n+1})_{tt} + \varepsilon P_{N_n} Df(\widetilde{u}_n) D_\omega \widetilde{u}_n - \varepsilon P_{N_{n+1}} Df(u_{n+1}) D_\omega \widetilde{u}_n . \tag{27}$$

We deduce from (24)-(27) the estimate $\|D_{\varepsilon,\omega} h_{n+1}\|_s \le K\gamma^{-1} N_{n+1}^{-1}$ using (11), (17), (18), $\|h_{n+1}\|_s \le \rho_{n+1}$ in lemma 2.1, and the definition of β in (16); we omit the details. ■

We now define, by means of a cut-off function, a C^1-extension of $h_{n+1} \in C^1(A_{n+1}, W^{(N_n)})$ onto the whole A. Note that h_{n+1} is yet defined on a neighborhood of A_{n+1} of width $\gamma N_{n+1}^{-\sigma}$ and $\|h_{n+1}\|_s = O(\varepsilon \gamma^{-1} N_{n+1}^{-\sigma-1})$.

Lemma 2.3 (Whitney extension) *There exists an extension $\widetilde{h}_{n+1} \in C^1(A, W^{(N_{n+1})})$ of h_{n+1} satisfying, for $\varepsilon\gamma^{-1}$ small, $\|\widetilde{h}_{n+1}\|_s \le K\varepsilon\gamma^{-1} N_{n+1}^{-\sigma-1}$, $\|D_{\varepsilon,\omega} \widetilde{h}_{n+1}\|_s \le K\gamma^{-1} N_{n+1}^{-1}$.*

PROOF. Let

$$\widetilde{h}_{n+1}(\varepsilon, \omega) := \begin{cases} \psi_{n+1}(\varepsilon, \omega) h_{n+1}(\varepsilon, \omega) & \text{if} \quad (\varepsilon, \omega) \in \mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma}) \\ 0 & \text{if} \quad (\varepsilon, \omega) \notin \mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma}) \end{cases} \tag{28}$$

where ψ_{n+1} is a C^∞ cut-off function satisfying $0 \le \psi_{n+1} \le 1$, $\psi_{n+1} \equiv 1$ on $\mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma}/2)$, $\psi_{n+1} \equiv 0$ outside $\mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma})$, and $|D_{\varepsilon,\omega} \psi_{n+1}| \le \gamma^{-1} N_{n+1}^{\sigma} C$ (a cut-off function ψ_{n+1} can be constructed like in lemma 3.3 of [3]).

Then $\|\widetilde{h}_{n+1}\|_s \le \|h_{n+1}\|_s \le K\varepsilon\gamma^{-1} N_{n+1}^{-\sigma-1}$ by lemma 2.1, and, for $\varepsilon\gamma^{-1} \le 1$,

$$\|D_{\varepsilon,\omega} \widetilde{h}_{n+1}\|_s \le |D_{\varepsilon,\omega} \psi_{n+1}| \, \|h_{n+1}\|_s + \|D_{\varepsilon,\omega} h_{n+1}\|_s \le \bar{K} \gamma^{-1} N_{n+1}^{-1}$$

thanks to the estimate $\|D_{\varepsilon,\omega} h_{n+1}\|_s \le K\gamma^{-1} N_{n+1}^{-1}$ proved in lemma 2.2. ■

Finally we define the C^1-Whitney extension $\widetilde{u}_{n+1} \in C^1(A, W^{(N_{n+1})})$ of u_{n+1} as

$$\widetilde{u}_{n+1} := \widetilde{u}_n + \widetilde{h}_{n+1} .$$

To complete the induction proof of Theorem 2.1 we have to prove that $\widetilde{u}_{n+1}$ and A_{n+1} satisfy all the properties stated in the Theorem plus (14)-(15) at the step $n+1$.

By lemma 2.3 property (14) is verified at the step $n+1$. Then we get also $\|\widetilde{u}_{n+1}\|_s \le K\varepsilon\gamma^{-1}$ and $\|D_{\varepsilon,\omega} \widetilde{u}_{n+1}\|_s \le K\gamma^{-1}$.

Now we prove that also (15) holds at the step $n+1$.

Lemma 2.4 *For $\varepsilon \le \varepsilon_0(\gamma, N_0, \beta)$ small enough,*

$$B_{n+1} \le (1 + N_{n+1}^{\mu}) B_n \qquad \text{and} \qquad B'_{n+1} \le B'_n + K\gamma^{-1} N_{n+1}^{\mu} (N_{n+1}^{\sigma} B_n + \varepsilon B'_n) .$$

PROOF. We have $B_{n+1} \le B_n + \|\widetilde{h}_{n+1}\|_{s+\beta}$ where, by (28), $\|\widetilde{h}_{n+1}\|_{s+\beta} \le \|h_{n+1}\|_{s+\beta}$ and, for all $(\varepsilon, \omega) \in \mathcal{N}(A_{n+1}, \gamma N_{n+1}^{-\sigma})$, $h_{n+1} = -\mathcal{L}_{N_{n+1}}^{-1}(\varepsilon, \omega, \widetilde{u}_n)(r_n + R_n(h_{n+1}))$. Then

$$\|h_{n+1}\|_{s+\beta} \overset{(P5)}{\le} K \frac{N_{n+1}^{\mu}}{\gamma} \Big(\|r_n\|_{s+\beta} + \|R_n(h_{n+1})\|_{s+\beta} + \|\widetilde{u}_n\|_{s+\beta} (\|r_n\|_s + \|R_n(h_{n+1})\|_s) \Big) . \tag{29}$$

By (19) and the tame estimate (P2),

$$\|r_n\|_{s+\beta} \le \varepsilon \|f(\widetilde{u}_n)\|_{s+\beta} \overset{(P2)}{\le} \varepsilon K (1 + \|\widetilde{u}_n\|_{s+\beta}) = \varepsilon K B_n . \tag{30}$$

By the Taylor tame estimate (P3), and since $\|h_{n+1}\|_s \le \rho_{n+1} = 2K_1 \varepsilon \gamma^{-1} N_{n+1}^{-\sigma-1}$ (lemma 2.1)

$$\|R_n(h_{n+1})\|_{s+\beta} \stackrel{(P3)}{\le} \varepsilon K\Big(\|\widetilde{u}_n\|_{s+\beta}\|h_{n+1}\|_s^2 + \|h_{n+1}\|_s\|h_{n+1}\|_{s+\beta}\Big) \le \varepsilon K\Big(B_n\rho_{n+1}^2 + \rho_{n+1}\|h_{n+1}\|_{s+\beta}\Big). \tag{31}$$

Inserting in (29) estimates (30)-(31), $\|r_n\|_s$, $\|R_n(h_{n+1})\|_s \le K\varepsilon$, we get

$$\|h_{n+1}\|_{s+\beta} \le \bar{K}\frac{\varepsilon}{\gamma}N_{n+1}^{\mu}B_n + \Big(\bar{K}\frac{\varepsilon}{\gamma}N_{n+1}^{\mu-\sigma-1}\Big)\|h_{n+1}\|_{s+\beta} \le \bar{K}\frac{\varepsilon}{\gamma}N_{n+1}^{\mu}B_n + \frac{1}{2}\|h_{n+1}\|_{s+\beta}$$

for $\bar{K}\varepsilon\gamma^{-1} < 1/2$. Hence $\|h_{n+1}\|_{s+\beta} \le N_{n+1}^{\mu}B_n$. The second inequality follows similarly by (25)-(27) and using $\|D_{\varepsilon,\omega}\widetilde{h}_{n+1}\|_{s+\beta} \le C\gamma^{-1}N_{n+1}^{\sigma}\|h_{n+1}\|_{s+\beta} + \|D_{\varepsilon,\omega}h_{n+1}\|_{s+\beta}$. ■

Lemma 2.5 (Measure estimate) $A_\infty := \bigcap_{n=0}^{\infty}$ *satisfies* $|A_\infty| \ge \varepsilon_0(\bar{\omega}_2 - \bar{\omega}_1) - C\varepsilon_0\gamma$.

PROOF. Recalling (20), the complementary $A_\infty^c = \bigcup_{n=0}^{\infty} G_{N_{n+1}}^c(\widetilde{u}_n)$ and, by (P5), $G_{N_{n+1}}^c(\widetilde{u}_n) = G^c$ for all n such that $N_{n+1} \le c(\gamma\varepsilon_0^{-1})^{2/3}$, that is all $n = 0, 1 \ldots, n^*$ for some n^*, so that $A_{n^*+1} = A_{n^*} = \ldots = A_0$. By standard arguments, $|G^c| \le C\varepsilon_0\gamma$. Since $\|\widetilde{u}_n - \widetilde{u}_{n-1}\| \le N_n^{-\sigma}$, $\forall n$, by (13),

$$\begin{aligned}\Big|\bigcup_{n=0}^{\infty} G_{N_{n+1}}^c(\widetilde{u}_n)\Big| &= \sum_{n>n^*}^{\infty}\Big|G_{N_{n+1}}^c(\widetilde{u}_n) \setminus G_{N_n}^c(\widetilde{u}_{n-1})\Big| + |G^c| \\ &\le \sum_{n>n^*} \varepsilon_0\gamma N_n^{-1} + |G^c| \le C'\varepsilon_0\gamma\end{aligned}$$

and we get the thesis. ■

Finally, by (14), we deduce that, $\forall(\varepsilon,\omega) \in A_\infty$, the series

$$\widetilde{u} := \widetilde{u}_0 + \sum_{n=1}^{\infty}(\widetilde{u}_n - \widetilde{u}_{n-1})$$

converges in H^s (uniformly in (ε,ω)) to a solution of equation (3) and $\|\widetilde{u}\|_s \le K\varepsilon\gamma^{-1}$, $\|D_{\varepsilon,\omega}\widetilde{u}\|_s \le K\gamma^{-1}$. The proof of Theorem 2.1 is complete. ■

3 The linearized problem: proof of (P5)

3.1 Preliminaries

For $A \subset \mathbf{Z}^{d+1}$, A finite and symmetric (i.e. $-A = A$), we define the finite dimensional subspace of H^s

$$H_A := \mathrm{Span}_{k\in A}\{e_k\} = \Big\{\sum_{k\in A} h_k e_k \ : \ h_k \in \mathbf{C},\ h_k^* = h_{-k}\Big\} \quad \text{where} \quad e_k(t,x) := e^{\mathrm{i}(lt + j\cdot x)}$$

and $k := (l,j) \in \mathbf{Z}\times\mathbf{Z}^d$. We shall denote by P_A the L^2-orthogonal projector on H_A, defined by

$$P_A h := \sum_{k\in A} h_k e_k\,, \qquad \forall h = \sum_{k\in\mathbf{Z}^{d+1}} h_k e_k \in H^s\,.$$

Note that P_A is also the H^s-orthogonal projector onto H_A. With these notations, the linear operator $\mathcal{L}_N$ in (10) is defined on H_{Ω_N}, where

$$\Omega_N := \Big\{k := (l,j) \in \mathbf{Z}\times\mathbf{Z}^d \ \Big|\ |(l,j)| \le N\Big\}$$

and we write[5]

$$\mathcal{L}_N = D + \varepsilon T \quad \text{with} \quad Dh := \omega^2 h_{tt} - \Delta h + mh \,, \quad Th := P_{\Omega_N}(a\, h) \,, \quad a(t,x) := -(\partial_u F)(t,x,u(t,x)) \,.$$

Note that T, as u, depends on the parameters ε and ω. In the L^2- orthonormal basis $(e_k)_{k\in\Omega_N}$ of H_{Ω_N}, D is represented by a diagonal matrix with eigenvalues

$$d_k = d_{(l,j)} := -\omega^2 l^2 + \lambda_j \,, \qquad \lambda_j := |j^2| + m \,,$$

whereas T is represented by the self-adjoint Toepliz matrix $(a_{k-k'})_{k,k'\in\Omega_N}$, the a_k being the Fourier coefficients of the function $a(t,x)$.

Definition 3.1 *Given $U \in \mathcal{L}(H_{\Omega_N}, H_{\Omega_N})$ and $A, B \subset \Omega_N$, we define the linear operator*

$$U_B^A : H_A \to H_B \,, \qquad U_B^A := P_B U|_{H_A}$$

and its operatorial norm $\|U_B^A\|_s := \sup\{\|U_B^A h\|_s,\ h \in H_A,\ \|h\|_s = 1\}$. We set

$$L_B^A := P_B \mathcal{L}_N|_{H_A} = P_B(D + \varepsilon T)|_{H_A} = D_B^A + \varepsilon T_B^A \,.$$

Note that L_A^B is the L^2-adjoint of L_B^A and, in particular, L_A^A is L^2-selfadjoint.

Lemma 3.1 *Let $U \in \mathcal{L}(H_A, H_A)$ be L^2-selfadjoint.*
i) Its eigenvalues λ_k satisfy $\min_k |\lambda_k| \geq \bar\lambda > 0$ if and only if $\|U^{-1}\|_0 \leq \bar\lambda^{-1}$.
ii) If true then, $\forall s' \geq 0$, $\|U^{-1}\|_{s'} \leq \bar\lambda^{-1}\Big(\frac{M_A}{m_A}\Big)^{s'}$ where $M_A := \max_{k\in A} |k|$, $m_A := \min_{k\in A} |k|$.

PROOF. $i)$ is standard. For $ii)$, by the smoothing estimates $(P4)$ and lemma 3.1 we get

$$\|U^{-1}h\|_{s'} \leq M_A^{s'} \|U^{-1}h\|_0 \leq \frac{M_A^{s'}}{\bar\lambda} \|h\|_0 \leq \frac{M_A^{s'}}{m_A^{s'}} \frac{1}{\bar\lambda} \|h\|_{s'} \,.$$

■

A bounded subset A of $\mathbf{Z}^{d+1}$ is said *dyadic* if $M_A \leq 2m_A$ where $M_A := \max_{k\in A}|k|$, $m_A := \min_{k\in A}|k|$. By lemma 3.1, the inverse of the minimum modulus of the eigenvalues of a self-adjoint operator U, acting on a subspace H_A with dyadic A, provides also a bound of the Sobolev operatorial norm $\|U^{-1}\|_{s'}$.

In the next lemma we estimate the variations of the eigenvalues of $L_A^A(\varepsilon,\omega)$ with respect to ω.

Lemma 3.2 *Let $A \subset \Omega_N$, $0 < \bar\omega_1 < \bar\omega_2$ and I be any compact interval in $[-\gamma,\gamma]$, of length $|I|$. Suppose $\|u\|_{C^1(s)} \leq C\gamma^{-1}$. There exists $\varepsilon_0(\gamma) > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0(\gamma)$,*

$$\Big|\{\omega \in [\bar\omega_1, \bar\omega_2] \text{ s.t. at least one eigenvalue of } L_A^A(\varepsilon,\omega) \text{ belongs to } I\}\Big| \leq \frac{|A|\,|I|}{\bar\omega_1} \tag{32}$$

where $|A|$ denotes the number of elements of A.

PROOF. Since the map $\omega \mapsto L_A^A(\varepsilon,\omega)$ is C^1 and each $L_A^A(\varepsilon,\omega)$ is selfadjoint, its eigenvalues can be listed as C^1 functions $\lambda_k(\varepsilon,\omega)$, $1 \leq k \leq |A|$, of ω. In what follows, $0 < \varepsilon \leq \varepsilon_0(\gamma)$ is fixed and we omit the dependency with respect to ε.

Denoting $E_{\omega,k}$ the eigenspace of $L_A^A(\omega)$ associated to $\lambda_k(\omega)$,

$$\begin{aligned}(\partial_\omega \lambda_k)(\omega) \leq \max_{h\in E_{\omega,k}, \|h\|_0=1} \Big((\partial_\omega L_A^A)(\omega)h, h\Big)_0 &= \max_{h\in E_{\omega,k}, \|h\|_0=1} \Big(2\omega(\partial_{tt}h, h)_0 + O(\varepsilon\gamma^{-1})\Big) \\ &\leq \max_{h\in E_{\omega,k}, \|h\|_0=1} \Big(-2\omega\|\partial_t h\|_0^2 + O(\varepsilon\gamma^{-1})\Big)\end{aligned} \tag{33}$$

[5] Comparing with the notations of section 2, $H_{\Omega_N} \equiv W^{(N)}$, $D \equiv L_\omega$ and $P_{\Omega_N} \equiv P_N$.

using that $\|\partial_\omega a\|_s = \|(\partial_u^2 F)(t,x,u)\,\partial_\omega u\|_s \le C\gamma^{-1}$ and integrating by parts.

Let P_1 denote the L^2-orthogonal projector on $H_{1,A} := \{h \in H_A \mid \partial_t h = 0\}$.

Claim: Let $\lambda_k(\omega) \in [-\gamma,\gamma]$. For $h \in E_{\omega,k}$, decompose $h = h_1 + h_2$ with $h_1 := P_1 h$, $h_2 := (I-P_1)h \in H_{1,A}^{\perp}$. Then, for ε_0 small enough, $\|h_2\|_0^2 \ge 3\|h\|_0^2/4$.

Indeed, if $h \in E_{\omega,k}$, $\omega^2 h_{tt} - \Delta h + mh + \varepsilon P_A(ah) = \lambda_k(\omega)h$ and so h_1 satisfies

$$-\Delta h_1 + mh_1 - \lambda_k(\omega)h_1 = -\varepsilon P_1 P_A(ah)\,. \tag{34}$$

Furthermore, by (2), $\||j|^2 + m| \ge 3\gamma$, $\forall j \in \mathbf{Z}^d$, and if $\lambda_k(\omega) \in [-\gamma,\gamma]$ then $\||j|^2 + m - \lambda_k(\omega)| \ge 2\gamma$. Hence $-\Delta + m - \lambda_k(\varepsilon,\omega)$ is a L^2-selfadjoint operator of H_A whose eigenvalues are of modulus $\ge 2\gamma$. From (34),

$$\|h_1\|_0 \le \frac{\varepsilon}{2\gamma}\|P_1 P_A(ah)\|_0 \le \frac{C\varepsilon}{\gamma}\|a\|_s\|h\|_0 \le \frac{\|h\|_0}{2}$$

for $\varepsilon\gamma^{-1}$ small enough, and the claim follows.

Finally, since $\|\partial_t h\|_0^2 = \|\partial_t h_2\|_0^2 \ge \|h_2\|_0^2$, we deduce, from (33) and the previous claim,

$$(\partial_\omega \lambda_k)(\omega) \le \max_{h \in E_{\omega,k}, \|h\|_0 = 1}\Big(-2\omega\|h_2\|_0^2 + O(\varepsilon\gamma^{-1})\Big) \le -\frac{3\omega}{2} + O(\varepsilon\gamma^{-1}) \le -\omega \le -\bar\omega_1$$

for $0 < \varepsilon \le \varepsilon_0(\gamma)$ small enough. Therefore, $|\lambda_k^{-1}(I) \cap (\bar\omega_1,\bar\omega_2)| \le |I|/\bar\omega_1$ and, summing over all the $|A|$ eigenvalues, we deduce (32). ■

3.2 Regular and Singular sites

Definition 3.2 *Fixed $\rho > 0$ we define the* **regular sites** *R and the* **singular sites** *S, as*

$$R := \Big\{k \in \Omega_N \mid |d_k| \ge \rho\Big\} \quad \text{and} \quad S := \Omega_N \backslash R := \Big\{k \in \Omega_N \mid |d_k| < \rho\Big\}\,.$$

We introduce the following assumption: the singular sites S can be partitioned in disjoint clusters Ω_α,

$$S = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha \tag{35}$$

satisfying:

- (H1) **(dyadic)** $M_\alpha \le 2m_\alpha$, $\forall \alpha \in \mathcal{A}$, where $M_\alpha := \max_{k \in \Omega_\alpha}|k|$, $m_\alpha := \min_{k\in\Omega_\alpha}|k|$.
- (H2) **(separation)** $\exists \delta := \delta(d) \in (0,1)$ such that $\mathrm{d}(\Omega_\alpha,\Omega_\sigma) := \min_{k\in\Omega_\alpha, k'\in\Omega_\sigma}|k-k'| \ge (M_\alpha + M_\sigma)^\delta/2$, $\forall \alpha \ne \sigma$.

For the wave equation (3), we shall verify both properties (H1)-(H2) in lemma 3.6 and section 4.

Remark 3.1 *Condition (H1) is weaker than the corresponding one in [7] (Lemma 7 of Appendix 2). It could be considerably weakened, but not completely eliminated, see remark 3.2.*

For $K \le N$ let

$$\Omega_K := \Big\{k \in \mathbf{Z}^{d+1} \mid |k| \le K\Big\} \qquad \text{and} \qquad L_K := L_{\Omega_K}^{\Omega_K}\,. \tag{36}$$

Proposition 3.1 *Assume that there is a partition of S satisfying (H1)-(H2). Given $\tau > 0$, there exist $\mu := \mu(d,\tau)$, $s := s(\delta,d,\tau)$ (independent of N) and, $\forall \gamma \in (0,1)$, $\forall s' \ge s$, there is $\varepsilon_0 := \varepsilon_0(\gamma,s') > 0$ such that : if $< \varepsilon \le \varepsilon_0$ and $\forall K \le N$, all the eigenvalues of L_K have modulus $\ge \gamma K^{-\tau}$ then*

$$\|\mathcal{L}_N^{-1}h\|_{s'} \le \frac{K(s')}{\gamma}N^\mu\Big(\|h\|_{s'} + \|u\|_{s'}\|h\|_s\Big)\,, \qquad \forall h \in H_{\Omega_N}\,. \tag{37}$$

Remark 3.2 *Without exploiting any separation property of the clusters we would only have an estimate like* $\forall N$, $\|\mathcal{L}_N^{-1}h\|_{s'} \le N^{s'}\|h\|_{s'}$, $\forall h \in H_{\Omega_N}$. *This is even weaker than a tame like estimate like* $\|\mathcal{L}_N^{-1}h\|_{s'} \le K\|h\|_{2s'}$. *In [19] Lojasiewicz and Zehnder have shown with a counter-example that a so weak condition is not enough for the convergence of the Nash-Moser iteration scheme, whereas a tame estimate like* $\|\mathcal{L}_N^{-1}h\|_{s'} \le K\|h\|_{\lambda s'}$ *with* $\lambda < 2$ *would be enough. This fact confirms that some more information, in addition to lower bounds for the moduli of the eigenvalues, is required.*

We shall deduce property (P5) from Proposition 3.1 in the next section. The proof of Proposition 3.1 is provided in sections 3.4, 3.5, 3.6.

3.3 Proof of (P5)

Let

$$B_K(u) := \Big\{(\varepsilon,\omega) \in [0,\varepsilon_0] \times [\bar\omega_1,\bar\omega_2] \mid \|L_K^{-1}\|_0 \le \frac{K^\tau}{2\gamma}\Big\} \quad \text{and} \quad G_N(u) := \bigcap_{K \le N} B_K(u) \cap G\,.$$

By lemma 3.1, $(\varepsilon,\omega) \in B_K(u)$ if and only if all the eigenvalues λ_k, $k = 1,\ldots,K$, of L_K have modulus greater or equal to $2\gamma/K^\tau$. Below, to emphasize the dependence of L_K with respect to parameters, we shall use the notations $L_K(\varepsilon,\omega)$.

Lemma 3.3 *Let* $\tau \ge 3/2$. *There is* $c > 0$ *such that* $\forall K \le N_{\varepsilon_0} := c(\gamma\varepsilon_0^{-1})^{2/3}$ *we have* $G \subset B_K(u)$. *As a consequence, if* $N \le N_{\varepsilon_0}$ *then* $G_N(u) = G$.

Proof. The eigenvalues of L_K have the form $-\omega^2 l^2 + |j|^2 + m + O(\varepsilon)$ with $|(l,j)| \le K$. Hence, $\forall(\varepsilon,\omega) \in G$, as a consequence of (2), if $\varepsilon_0\gamma^{-1}K^{3/2}$ is small enough, then all the eigenvalues of $L_K(\varepsilon,\omega)$ have modulus $\ge 2\gamma/K^{3/2}$. ■

On the other hand, for $N > N_{\varepsilon_0} := c(\gamma\varepsilon_0^{-1})^{2/3}$, we have to excise additional "resonant parameters".

Lemma 3.4 *Let* $N > N_{\varepsilon_0}$. *If* $(\varepsilon,\omega) \in \mathcal{N}(G_N(u), \gamma N^{-3-\tau})$ *then,* $\forall K \le N$, *all the eigenvalues of* $L_K(\varepsilon,\omega)$ *have modulus* $\ge \gamma K^{-\tau}$.

Proof. For $\omega,\omega' \in [\bar\omega_1,\bar\omega_2]$ and $\varepsilon,\varepsilon' \in [0,\varepsilon_0]$, we have, $\forall h \in H_{\Omega_K}$,

$$\begin{aligned}\Big\|(L_K(\varepsilon,\omega) - L_K(\varepsilon',\omega'))h\Big\|_0 &\le |\omega^2 - \omega'^2|\|h_{tt}\|_0 + \varepsilon'\|a(\varepsilon,\omega) - a(\varepsilon',\omega')\|_s\|h\|_0 + |\varepsilon-\varepsilon'|\|a(\varepsilon,\omega)\|_s\|h\|_0 \\ &\le CK^2(|\omega-\omega'| + |\varepsilon-\varepsilon'|)\|h\|_0\,.\end{aligned}$$

If $(\varepsilon,\omega) \in \mathcal{N}(G_N(u),\gamma N^{-3-\tau})$, there is $(\varepsilon',\omega') \in G_N(u)$ such that $|\omega-\omega'| + |\varepsilon-\varepsilon'| \le \gamma N^{-3-\tau}$ and $\forall K \le N$, $\forall h \in H_{\Omega_K}$,

$$\|L_K(\varepsilon,\omega)h\|_0 \ge \|L_K(\varepsilon',\omega')h\|_0 - C\frac{\gamma K^2}{N^{3+\tau}}\|h\|_0 \ge \Big(\frac{2\gamma}{K^\tau} - C\frac{\gamma K^2}{N^{3+\tau}}\Big)\|h\|_0 \ge \frac{\gamma}{K^\tau}\|h\|_0\,,$$

by the definition of $B_K(u)$. The result follows. ■

Lemma 3.5 *Let* $\sigma,\tau \ge d+3$. *Then the measure estimate (13) holds.*

Proof. For $N' \ge N$,

$$G^c_{N'}(u_2)\backslash G^c_N(u_1) = G^c_{N'}(u_2) \cap G_N(u_1) \subset \Big[\cup_{K\le N}\,(B^c_K(u_2) \cap B_K(u_1) \cap G)\Big] \cup \Big[\cup_{K>N}\, B^c_K(u_2) \cap G\Big]\,.$$

By lemma 3.3, if $K \le N_{\varepsilon_0}$ then $B^c_K(u_2) \cap G = \emptyset$. Hence, to show (13), it is enough to prove that, if $\|u_1 - u_2\|_s \le N^{-\sigma}$, then

$$\sum_{K\le N} |B^c_K(u_2) \cap B_K(u_1)| + \sum_{K>\max(N,N_{\varepsilon_0})} |B^c_K(u_2)| \le \bar C\frac{\gamma\varepsilon_0}{N}\,.$$

Since $\|L_K(u_2) - L_K(u_1)\|_0 = O(\varepsilon\|u_2 - u_1\|_s) = O(\varepsilon N^{-\sigma})$, if one of the eigenvalues of $L_K(u_2)$ is in $[-2\gamma K^{-\tau}, 2\gamma K^{-\tau}]$ then, by the variational characterization of the eigenvalues of a selfadjoint operator, one of the eigenvalues of $L_K(u_1)$ is in $[-2\gamma K^{-\tau} - C\varepsilon N^{-\sigma}, 2\gamma K^{-\tau} + C\varepsilon N^{-\sigma}]$. As a result

$$\begin{aligned} B_K^c(u_2) \cap B_K(u_1) \quad \subset \quad & \{(\varepsilon, \omega) \mid \exists \text{ at least one eigenvalue of } L_K(u_1) \\ & \text{with modulus in } [2\gamma K^{-\tau}, 2\gamma K^{-\tau} + C\varepsilon N^{-\sigma}]\} . \end{aligned}$$

Then, by lemma 3.2, $|\{\omega|(\varepsilon,\omega) \in B_K^c(u_2) \cap B_K(u_1)\}| \leq C\varepsilon N^{-\sigma}|\Omega_K|/\bar{\omega}_1$ for each $\varepsilon \in (0, \varepsilon_0]$, whence

$$|B_K^c(u_2) \cap B_K(u_1)| \leq C'\varepsilon_0^2|\Omega_K|N^{-\sigma} \leq C'\varepsilon_0^2 K^{d+1}N^{-\sigma} .$$

Moreover, still by lemma 3.2, $|B_K^c(u_2)| \leq C\varepsilon_0|\Omega_K|\gamma K^{-\tau}/\bar{\omega}_1 \leq C'\varepsilon_0\gamma K^{d+1-\tau}$. Hence, for $\sigma, \tau \geq d+3$,

$$\begin{aligned} & \sum_{K \leq N} |B_K^c(u_2) \cap B_K(u_1)| + \sum_{K > \max(N, N_{\varepsilon_0})} |B_K^c(u_2)| \\ \leq \quad & C\varepsilon_0^2\Big(\sum_{K\leq N} K^{d+1}\Big)N^{-\sigma} + C\varepsilon_0\gamma\Big(\sum_{K>\max(N,N_{\varepsilon_0})} K^{d+1-\tau}\Big) \\ \leq \quad & C\varepsilon_0^2 N^{d+2-\sigma} + C'\varepsilon_0\gamma(\max(N, N_{\varepsilon_0}))^{d+2-\tau} \leq \bar{C}\gamma\varepsilon_0 N^{-1}, \end{aligned}$$

provided that ε_0 is small enough. ■

Lemma 3.6 (Separation in clusters) $\forall\,\gamma \in (0,1)$ *let* $\rho(\gamma) := \gamma^{3r(d)+1}/8$ *as defined in lemma 4.4. There is* $\varepsilon_0(\gamma)$ *such that* $\forall N \geq N_{\varepsilon_0} := c(\gamma\varepsilon_0^{-1})^{2/3}$, $\forall\,(\varepsilon,\omega) \in \mathcal{N}(G, \gamma N^{-\sigma})$, $\sigma \geq 3$, *there exists a decomposition of the singular sites*

$$S := \Big\{(l,j) \in \Omega_N \mid |\omega^2 l^2 - \lambda_j| < \rho := \frac{\rho(\gamma)}{2}\Big\} = \bigcup_{\alpha\in\mathcal{A}} \Omega_\alpha \tag{38}$$

in clusters, satisfying both properties (H1)-(H2) for some $\delta := \delta(d)$.

PROOF. Let $(\varepsilon, \omega') \in G$. By lemma 4.4, $\forall\gamma \in (0,1)$,

$$S' := \Big\{(l,j) \in \Omega_N \mid |(\omega')^2 l^2 - \lambda_j| < \rho(\gamma)\Big\} = \bigcup_{\alpha\in\mathcal{A}} \Omega'_\alpha$$

satisfying (H1)-(H2) for some $\delta := \delta(d)$. If $\varepsilon_0(\gamma)$ is small enough, then $\forall N \geq N_{\varepsilon_0}$, $\gamma N^{-\sigma} \leq \rho(\gamma)N^{-2}/4\bar{\omega}_2$, and, $\forall|\omega - \omega'| \leq \gamma N^{-\sigma}$, we have $S \subset S'$. Therefore $S = \bigcup_{\alpha\in\mathcal{A}} \Omega_\alpha$ where $\Omega_\alpha := \Omega'_\alpha \cap S$ satisfy (H1)-(H2). ■

PROOF OF PROPERTY (P5) COMPLETED. Let $(\varepsilon,\omega) \in \mathcal{N}(G_N, \gamma N^{-\sigma})$ with $\sigma \geq 3+\tau$. Then, by lemma 3.6, there exists a partition of the singular sites S satisfying assumptions (H1)-(H2). Furthermore, by lemma 3.4, $\forall K \leq N$, all the eigenvalues of L_K have modulus $\geq \gamma K^{-\tau}$. Therefore, by Proposition 3.1, $\mathcal{L}_N(\varepsilon,\omega,u)$ is invertible and (11) holds (we can fix $\tau = d+3$ by lemma 3.5).

The measure estimate (13) is proved in lemma 3.5.

The remainder of this section is devoted to the proof of Proposition 3.1.

3.4 Reduction along the regular sites

We have to solve the linear system

$$\mathcal{L}_N h = b\,, \qquad h, b \in H_{\Omega_N} . \tag{39}$$

According to the splitting of the indexes $\Omega_N = R \cup S$ we decompose

$$H_{\Omega_N} = H_R \oplus H_S$$

in orthogonal subspaces. Writing the unique decomposition $h = h_R + h_S$, $b = b_R + b_S$, with $h_R, b_R \in H_R$, $h_S, b_S \in H_S$, problem (39) is equivalent to

$$\begin{cases} L_R^R h_R + L_R^S h_S = b_R \\ L_S^R h_R + L_S^S h_S = b_S \,. \end{cases} \tag{40}$$

We recall that $s > (d+1)/2$ so that H^s is a Banach algebra.

To prove the invertibility of $L_R^R = D_R + \varepsilon T_R$ where, for brevity, $D_R := D_R^R$, $T_R := T_R^R$, note that

$$\forall s \geq 0\,, \quad \|D_R h\|_s \geq (\min_{k \in R} |d_k|)\|h\|_s \geq \rho \|h\|_s\,, \quad \forall h \in H_R\,. \tag{41}$$

Lemma 3.7 *There is $c > 0$ such that if $\varepsilon\|a\|_s/\rho \leq c$ then $\|(L_R^R)^{-1}\|_0, \|(L_R^R)^{-1}\|_s \leq 2\rho^{-1}$.*

Proof. We have, $\forall h \in H_R$,

$$\|L_R^R h\|_s \geq \|D_R h\|_s - \varepsilon\|T_R h\|_s \geq (\rho - \varepsilon C\|a\|_s)\|h\|_s$$

and the result follows. Similarly for $\|(L_R^R)^{-1}\|_0$. ■

Next we estimate the Sobolev norm $\|(L_R^R)^{-1}h\|_{s'}$ for $s' \geq s$. Since T comes from the multiplication operator for the function $a \in H^s$, it is natural to expect the following interpolation type inequality.

Lemma 3.8 *For $\varepsilon(\|a\|_s + 1)/\rho \leq c(s')$ small enough, $\|(L_R^R)^{-1}h\|_{s'} \leq 2\rho^{-1}(\|a\|_{s'}\,\|h\|_s + \|h\|_{s'})$.*

Proof For $h \in H_R$ let $v := (L_R^R)^{-1}h$. Then

$$\|h\|_{s'} = \|D_R v + \varepsilon T_R v\|_{s'} \geq \|D_R v\|_{s'} - \varepsilon\|T_R v\|_{s'} \overset{(41)}{\geq} \rho\|v\|_{s'} - \varepsilon\|T_R v\|_{s'}\,.$$

Now $\|T_R h\|_{s'} \leq C(s')(\|a\|_{s'}\|h\|_s + \|a\|_s\|h\|_{s'})$, $\forall h \in H_R$, (see e.g. [24]), hence

$$\|h\|_{s'} \geq (\rho - \varepsilon C(s')\|a\|_s)\|v\|_{s'} - \varepsilon C(s')\|a\|_{s'}\|v\|_s\,.$$

Therefore, if $\varepsilon C(s')(\|a\|_s + 1)\rho^{-1} \leq 1/2$,

$$\|v\|_{s'} \leq \frac{2}{\rho}\Big(\|h\|_{s'} + \varepsilon C(s')\|a\|_{s'}\|v\|_s\Big) \leq \frac{2}{\rho}\Big(\|h\|_{s'} + \|a\|_{s'}\|h\|_s\Big)\,,$$

by Lemma 3.7. ■

Lemma 3.9 $\forall A, B \subset \Omega_N$, $\forall s' \geq s$, $\|T_B^A\|_0 \leq \dfrac{C(s')\|a\|_{s'}}{(1 + \mathrm{d}(A,B))^{s'-(d+1)/2}}$.

Proof Using Hölder inequality and exchanging the order of integration

$$\begin{aligned} \|T_B^A h\|_0^2 &= \sum_{k \in B}\Big|\sum_{m \in A} a_{k-m}h_m\Big|^2 \leq \sum_{k \in B}\Big(\sum_{m \in A} |a_{k-m}|^2\,|h_m|^2(1 + |k-m|^{2s'})\Big)\Big(\sum_{m \in A}\frac{1}{1 + |k-m|^{2s'}}\Big) \\ &\leq \sum_{m \in A}|h_m|^2\Big(\sum_{k \in B}|a_{k-m}|^2\,(1 + |k-m|^{2s'})\Big)\sum_{|m| \geq \Delta}\frac{1}{1 + |m|^{2s'}} \leq \frac{C(s')\|h\|_0^2\,\|a\|_{s'}^2}{(1+\Delta)^{2s'-(d+1)}} \end{aligned}$$

where $\Delta := \mathrm{d}(A,B)$. ■

Lemma 3.10 *Let $U \in \mathcal{L}(H_R, H_R)$ be such that, for some $\kappa > 0$, $\forall A, B \subset R$,*

$$\|U_B^A\|_0 \leq \frac{C_1}{(1 + \mathrm{d}(A,B))^\kappa}\,.$$

Then, $\forall p \geq 1$, $\forall A, B \subset R$,

$$\Big\|(U^p)_B^A\Big\|_0 \leq \frac{C_1 p^{\kappa+1}\|U\|_0^{p-1}}{(1 + \mathrm{d}(A,B))^\kappa}\,. \tag{42}$$

Proof We proceed by induction noting that, for $p = 1$, (42) is just our assumption. Now assume that (42) holds up to order p. Define $\Delta := \mathrm{d}(A, B)$ and, for some $\lambda \in (0, 1)$ to be specified later, $B' = \mathcal{N}(B, \lambda\Delta) \cap R$, $B'' = R \backslash B'$. We have

$$\begin{aligned} \left\| (U^{p+1})_B^A \right\|_0 &= \left\| (U^p)_B^{B'} U_{B'}^A + (U^p)_B^{B''} U_{B''}^A \right\|_0 \le \|U^p\|_0 \|U_{B'}^A\|_0 + \|(U^p)_B^{B''}\|_0 \|U\|_0 \\ &\le \|U\|_0^p \frac{C_1}{(1 + \mathrm{d}(A, B'))^\kappa} + \frac{C_1 p^{\kappa+1} \|U\|_0^{p-1}}{(1 + \mathrm{d}(B'', B))^\kappa} \|U\|_0 \end{aligned}$$

by our hypothesis of induction. Now $\mathrm{d}(A, B') \ge (1 - \lambda)\Delta$ and $\mathrm{d}(B'', B) \ge \lambda\Delta$. Hence

$$\left\| (U^{p+1})_B^A \right\|_0 \le C_1 \|U\|_0^p \Big(\frac{1}{(1 + (1 - \lambda)\Delta)^\kappa} + \frac{p^{\kappa+1}}{(1 + \lambda\Delta)^\kappa} \Big) \le \frac{C_1 \|U\|_0^p}{(1 + \Delta)^\kappa} \Big(\frac{1}{(1 - \lambda)^\kappa} + \frac{p^{\kappa+1}}{\lambda^\kappa} \Big) .$$

Since $\min_{\lambda \in (0,1)} [(1 - \lambda)^{-\kappa} + p^{\kappa+1} \lambda^{-\kappa}] = (p + 1)^{\kappa+1}$ we have proved (42) up to order $p + 1$. ■

Lemma 3.11 *$\forall A, B \subset R$ with $A \cap B = \emptyset$, and for $\varepsilon \|a\|_s \rho^{-1} \le c$ small enough,*

$$\left\| [(L_R^R)^{-1}]_B^A \right\|_0 \le \frac{\varepsilon C(s') \|a\|_{s'}}{\rho^2 \, \mathrm{d}(A, B)^{s' - (d+1)/2}} .$$

Proof By the Newmann series expansion

$$(L_R^R)^{-1} := (D_R + \varepsilon T_R)^{-1} = \Big(I + \sum_{p \ge 1} (-1)^p (\varepsilon D_R^{-1} T_R)^p \Big) D_R^{-1} \tag{43}$$

and since $A \cap B = \emptyset$, $[(L_R^R)^{-1}]_B^A = \sum_{p \ge 1} (-1)^p \varepsilon^p (U^p)_B^A [D_R^{-1}]_A^A$ where $U := D_R^{-1} T_R$. Since $\|D_R^{-1}\|_0 \le \rho^{-1}$,

$$\left\| [(L_R^R)^{-1}]_B^A \right\|_0 \le \frac{1}{\rho} \sum_{p \ge 1} \varepsilon^p \|(U^p)_B^A\|_0 . \tag{44}$$

Now, by lemma 3.9, $\forall\, A, B \subset R$,

$$\|U_B^A\|_0 = \|[D_R^{-1}]_B^B T_B^A\|_0 \le \frac{1}{\rho} \|T_B^A\|_0 \le \frac{C(s') \|a\|_{s'}}{\rho (1 + \mathrm{d}(A, B))^\kappa}$$

with $\kappa := s' - (d + 1)/2$, whence, by lemma 3.10,

$$\|(U^p)_B^A\|_0 \le \frac{C(s') \|a\|_{s'} p^{\kappa+1} \|U\|_0^{p-1}}{\rho (1 + \mathrm{d}(A, B))^\kappa} \le \frac{C(s') \|a\|_{s'} p^{\kappa+1} \|a\|_s^{p-1}}{\rho^p \mathrm{d}(A, B)^\kappa}$$

using that $\|U\|_0 = \|D_R^{-1} T_R\|_0 \le \rho^{-1} \|a\|_s$. By (44) we get

$$\left\| [(L_R^R)^{-1}]_B^A \right\|_0 \le \frac{\varepsilon C(s') \|a\|_{s'}}{\rho^2 \, \mathrm{d}(A, B)^\kappa} \Big(\sum_{p \ge 1} \Big[\frac{\varepsilon \|a\|_s}{\rho} \Big]^{p-1} p^{\kappa+1} \Big)$$

proving the lemma. ■

Solving the first equation in (40) gives

$$h_R = (L_R^R)^{-1} (b_R - L_R^S h_S) \tag{45}$$

and inserting into the second equation in (40) gives

$$\Big[L_S^S - L_S^R (L_R^R)^{-1} L_R^S \Big] h_S = b_S - L_S^R (L_R^R)^{-1} b_R . \tag{46}$$

Our main task is to invert the selfadjoint operator

$$U : H_S \to H_S \,, \qquad U := L_S^S - L_S^R (L_R^R)^{-1} L_R^S \,. \tag{47}$$

According to (35) we get the orthogonal decomposition $H_S = \oplus_{\alpha\in\mathcal{A}} H_{\Omega_\alpha}$ which induces a block decomposition for the operator

$$U : \oplus_{\alpha\in\mathcal{A}} H_{\Omega_\alpha} \to \oplus_{\alpha\in\mathcal{A}} H_{\Omega_\alpha} \,, \quad U = \Big(U_{\Omega_\sigma}^{\Omega_\alpha} \Big)_{\alpha,\sigma\in\mathcal{A}} \,, \quad U_{\Omega_\sigma}^{\Omega_\alpha} = L_{\Omega_\sigma}^{\Omega_\alpha} - L_{\Omega_\sigma}^R (L_R^R)^{-1} L_R^{\Omega_\alpha}$$

where α is the column index and σ is the row index. We write

$$U = \mathcal{D} + \mathcal{R} \qquad \text{with} \qquad \mathcal{D} := \mathrm{diag}_{\alpha\in\mathcal{A}} (U_{\Omega_\alpha}^{\Omega_\alpha}) \quad \text{and} \quad \mathcal{R} := (U_{\Omega_\sigma}^{\Omega_\alpha})_{\alpha\neq\sigma} \,.$$

3.5 Estimates of the blocks

We always assume in the sequel that $\varepsilon \|a\|_s / \rho < c$. We first estimate the off-diagonal blocks $U_{\Omega_\sigma}^{\Omega_\alpha}$, $\alpha \neq \sigma$.

Lemma 3.12 *$\forall s' \geq s$, $\forall \alpha \neq \sigma$*

$$\|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0 \leq \frac{\varepsilon\, C(s') \|a\|_{s'}}{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)^{s' - \frac{d+1}{2}}} \,. \tag{48}$$

PROOF. The term $L_{\Omega_\sigma}^{\Omega_\alpha} = \varepsilon T_{\Omega_\alpha}^{\Omega_\sigma}$ satisfies estimate (48) by lemma 3.9. To estimate the term $L_{\Omega_\sigma}^R (L_R^R)^{-1} L_R^{\Omega_\alpha}$ we decompose $R = R_1 \cup R_2 \cup R_3$ where

$$R_1 := \Big\{ k \in R \mid \mathrm{d}(\Omega_\alpha, k) < \frac{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)}{3} \Big\},\ R_3 := \Big\{ k \in R \mid \mathrm{d}(\Omega_\sigma, k) < \frac{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)}{3} \Big\},\ R_2 := R \setminus (R_1 \cup R_3) \,.$$

Accordingly we have the decomposition $H_R = H_{R_1} \oplus H_{R_2} \oplus H_{R_3}$ and so

$$L_{\Omega_\sigma}^R (L_R^R)^{-1} L_R^{\Omega_\alpha} = \sum_{i=1}^3 \sum_{j=1}^3 L_{\Omega_\sigma}^{R_i} \Big[(L_R^R)^{-1} \Big]_{R_i}^{R_j} L_{R_j}^{\Omega_\alpha} \,.$$

To bound each term in the sum above, we distinguish three cases.
First case: $j \geq 2$. In this case $\mathrm{d}(R_j, \Omega_\alpha) \geq \mathrm{d}(\Omega_\alpha, \Omega_\sigma)/3$. Then, since $L_{R_j}^{\Omega_\alpha} = \varepsilon T_{R_j}^{\Omega_\alpha}$, by lemma 3.9,

$$\|L_{R_j}^{\Omega_\alpha}\|_0 \leq \frac{\varepsilon\, C(s') \|a\|_{s'}}{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)^{s' - \frac{d+1}{2}}} \,.$$

Using also $\|L_{\Omega_\sigma}^{R_i}\|_0 = \varepsilon \|T_{\Omega_\sigma}^{R_i}\|_0 \leq \varepsilon \|a\|_s$ and lemma 3.7, we get

$$\Big\| L_{\Omega_\sigma}^{R_i} \Big[(L_R^R)^{-1} \Big]_{R_i}^{R_j} L_{R_j}^{\Omega_\alpha} \Big\|_0 \leq \frac{2\varepsilon \|a\|_s}{\rho} \frac{\varepsilon C(s') \|a\|_{s'}}{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)^{s' - \frac{d+1}{2}}} \leq \frac{\varepsilon C(s') \|a\|_{s'}}{\mathrm{d}(\Omega_\alpha, \Omega_\sigma)^{s' - \frac{d+1}{2}}}$$

Second case: $j = 1$, $i = 1, 2$. Now $\mathrm{d}(R_i, \Omega_\sigma) \geq \mathrm{d}(\Omega_\alpha, \Omega_\sigma)/3$, and we proceed as in the previous case (the small factor is $\|L_{\Omega_\sigma}^{R_i}\|_0$).
Third case: $j = 1$, $i = 3$. Now $\mathrm{d}(R_1, R_3) \geq \mathrm{d}(\Omega_\alpha, \Omega_\sigma)/3$, and, by lemma 3.11,

$$\Big\| \Big[(L_R^R)^{-1} \Big]_{R_3}^{R_1} \Big\|_0 \leq \frac{\varepsilon C(s') \|a\|_{s'}}{\rho^2\, \mathrm{d}(\Omega_\alpha, \Omega_\sigma)^{s' - \frac{d+1}{2}}} \,.$$

Using also $\|L_{\Omega_\sigma}^{R_i}\|_0$, $\|L_{\Omega_\sigma}^{R_i}\|_0 \leq \varepsilon \|a\|_s$, we get estimate (48). ■

In the sequel we assume

$$s \geq s(d, \tau) := \frac{d+1}{2} + \frac{d+1+\tau}{\delta} + 1 \,, \tag{49}$$

where $\delta > 0$ is defined in $(H2)$ and $\tau > 0$ is the parameter introduced in Proposition 3.1.

Lemma 3.13 *Under the assumptions of Proposition 3.1,* $\forall\, 0 < \varepsilon \le \varepsilon_0(\gamma)$ *small enough,* $U_{\Omega_\alpha}^{\Omega_\alpha}$ *is invertible and* $\|(U_{\Omega_\alpha}^{\Omega_\alpha})^{-1}\|_0 \le c\gamma^{-1} M_\alpha^\tau$ *with* $c := 2^{\tau+2}$.

Proof It is sufficient to prove that

$$\|U_{\Omega_\alpha}^{\Omega_\alpha} w\|_0 \ge \frac{\gamma}{cM_\alpha^\tau}\|w\|_0\,, \qquad \forall w \in H_{\Omega_\alpha}\,. \tag{50}$$

For all $w \in H_{\Omega_\alpha} \subset H_S$, we have, recalling (47),

$$U_{\Omega_\alpha}^{\Omega_\alpha} w + \sum_{\sigma\neq\alpha} U_{\Omega_\sigma}^{\Omega_\alpha} w = Uw = (L_R^S + L_S^S)w - (L_R^R + L_S^R)(L_R^R)^{-1} L_R^S w = \mathcal{L}_N h \tag{51}$$

where $h := w - (L_R^R)^{-1} L_R^S w$.

Step 1: $\sum_{\sigma\neq\alpha} \|U_{\Omega_\sigma}^{\Omega_\alpha} w\|_0 \le \varepsilon \|a\|_s C(s) M_\alpha^{d+1-\nu}\|w\|_0$ where $\nu := \delta(s-(d+1)/2) > 0$.

Indeed, by lemma 3.12 (with $s' = s$)

$$\sum_{\sigma\neq\alpha} \|U_{\Omega_\sigma}^{\Omega_\alpha} w\|_0 \le \sum_{\sigma\neq\alpha} \frac{C\varepsilon\|a\|_s\|w\|_0}{\mathrm{d}(\Omega_\alpha,\Omega_\sigma)^{s-\frac{d+1}{2}}} \overset{(H2)}{\le} C\varepsilon\|a\|_s\|w\|_0 \sum_{\sigma\neq\alpha}\frac{1}{(M_\alpha+M_\sigma)^\nu}\,. \tag{52}$$

Next, for each $x \in \Omega_\sigma \subset S$, we define $M(x) := M_\sigma$ and $N(x) := |\Omega_\sigma|$. Then

$$\sum_{\sigma\neq\alpha}\frac{1}{(M_\alpha+M_\sigma)^\nu} = \sum_{x\in S\setminus\Omega_\alpha}\frac{1}{N(x)(M_\alpha+M(x))^\nu} \le \sum_{x\in\mathbf{Z}^{d+1},|x|\le N}\frac{1}{(M_\alpha+|x|)^\nu}$$

because $N(x)\ge 1$ and $M(x)\ge |x|$. Hence, for $\nu > d+1$,

$$\sum_{\sigma\neq\alpha}\frac{1}{(M_\alpha+M_\sigma)^\nu} \le C\int_1^{+\infty}\frac{r^d\,dr}{(M_\alpha+r)^\nu} \le \frac{C(\nu)}{(1+M_\alpha)^{\nu-d-1}}$$

proving, by (52), Step 1.

Step 2: $\|\mathcal{L}_N h\|_0 \ge 2^{-\tau-1}\gamma M_\alpha^{-\tau}\|w\|_0$.

Decompose $h = h' + h''$ with $h' := P_{\Omega_K} h$, $h'' := P_{\Omega_K^c} h$ and $K := 2M_\alpha$ (recall the notation in (36)). We have

$$\|\mathcal{L}_N h\|_0 \ge \|P_{\Omega_K}\mathcal{L}_N h\|_0 \ge \|L_K h'\|_0 - \varepsilon\|T_{\Omega_K}^{\Omega_K^c} h''\|_0 \ge \frac{\gamma}{(2M_\alpha)^\tau}\|h'\|_0 - \varepsilon C\|a\|_s\|h''\|_0, \tag{53}$$

by the assumptions on the eigenvalues of L_K in Proposition 3.1. Moreover since $h = w - (L_R^R)^{-1}L_R^S w$ and $w \in H_{\Omega_\alpha} \subset H_{\Omega_K}$,

$$h'' = -P_{\Omega_K^c}(L_R^R)^{-1}L_R^S w = -[(L_R^R)^{-1}]_{R\cap\Omega_K^c}^R L_R^{\Omega_\alpha} w.$$

Now $\mathrm{d}(\Omega_\alpha, R\cap\Omega_K^c) \ge M_\alpha$, from which we derive, arguing as in the proof of lemma 3.12, that

$$\|h''\|_0 \le \frac{C\varepsilon\|a\|_s}{\rho M_\alpha^{s-(d+1)/2}}\|w\|_0\,. \tag{54}$$

Furthermore, since $w \in H_{\Omega_\alpha}$ and $h' - w = -P_{\Omega_{2M_\alpha}}(L_R^R)^{-1}L_R^S w \in H_R$, we have $\|h'\|_0 = (\|h'-w\|_0^2 + \|w\|_0^2)^{1/2} \ge \|w\|_0$ and, (53), (54), imply

$$\|\mathcal{L}_N h\|_0 \ge \frac{\gamma\|w\|_0}{(2M_\alpha)^\tau} - \frac{C\varepsilon^2\|a\|_s^2}{\rho M_\alpha^{s-(d+1)/2}}\|w\|_0 \ge \frac{\gamma\|w\|_0}{(2M_\alpha)^\tau}\Big(1 - \frac{C'\varepsilon^2\|a\|_s^2}{\gamma\rho M_\alpha^{s-\tau-(d+1)/2}}\Big) \ge 2^{-\tau-1}\frac{\gamma}{M_\alpha^\tau}\|w\|_0$$

because $s > \tau + (d+1)/2$, and provided that $\varepsilon\gamma^{-1}$ is small enough.

Noting that $\nu > d+1-\tau$ (because $s \ge s(d,\tau)$), we deduce (50) from (51), Step 1 and Step 2, for $\varepsilon_0(\gamma)$ small enough. ■

3.6 Inversion of U and proof of Proposition 3.1

By lemma 3.13 the operator $\mathcal{D} := \mathrm{diag}_{\alpha\in\mathcal{A}}(U_{\Omega_\alpha}^{\Omega_\alpha})$ is invertible, and we write

$$U = \mathcal{D} + \mathcal{R} = \mathcal{D}(I + \mathcal{D}^{-1}\mathcal{R}) .$$

For the presence of the "small divisors", $\mathcal{D}^{-1}$ acts as an unbounded operator.

Lemma 3.14 $\forall s' \geq 0$, $\|\mathcal{D}^{-1}h\|_{s'} \leq K(s')\gamma^{-1}N^\tau \|h\|_{s'}$, $\forall h \in H_{\Omega_N}$.

Proof. For $h = \sum_{\alpha\in\mathcal{A}} h_\alpha$, $h_\alpha \in H_{\Omega_\alpha}$,

$$\|\mathcal{D}^{-1}h\|_{s'}^2 = \sum_{\alpha\in\mathcal{A}} \|(U_{\Omega_\alpha}^{\Omega_\alpha})^{-1}h_\alpha\|_{s'}^2 \overset{(P4)}{\leq} \sum_\alpha M_\alpha^{2s'} \|(U_{\Omega_\alpha}^{\Omega_\alpha})^{-1}h_\alpha\|_0^2 \leq \sum_\alpha M_\alpha^{2s'} \frac{cM_\alpha^{2\tau}}{\gamma^2} \|h_\alpha\|_0^2$$

by lemma 3.13. Then

$$\|\mathcal{D}^{-1}h\|_{s'}^2 \overset{(P4)}{\leq} \sum_{\alpha\in\mathcal{A}} \frac{cM_\alpha^{2\tau}}{\gamma^2} M_\alpha^{2s'} \frac{\|h_\alpha\|_{s'}^2}{m_\alpha^{2s'}} \overset{(H1)}{\leq} \frac{c2^{2s'}}{\gamma^2} \sum_{\alpha\in\mathcal{A}} M_\alpha^{2\tau}\|h_\alpha\|_{s'}^2 \leq \frac{C(s')}{\gamma^2} N^{2\tau}\|h\|_{s'}^2$$

because $M_\alpha \leq N$. ■

By lemma 3.12 the operator $\mathcal{R}$ acts somehow as a multiplication operator. Nevertheless, using the separation property (H2) we prove that $\mathcal{D}^{-1}\mathcal{R}$ is L^2-bounded and, using also the dyadic property (H1), we prove an interpolation type estimate for $\mathcal{D}^{-1}\mathcal{R}$ in high Sobolev norm.

Lemma 3.15 *Assume (49). Then*

$$\|\mathcal{D}^{-1}\mathcal{R}h\|_0 \leq K\frac{\varepsilon}{\gamma}\|a\|_s\|h\|_0 , \tag{55}$$

and, $\forall s' \geq s(d,\tau)$, *setting* $\mu_0 := \tau + 3(d+1)/2 > 0$,

$$\|\mathcal{D}^{-1}\mathcal{R}h\|_{s'} \leq K(s')\frac{\varepsilon}{\gamma}\Big(\|h\|_{s'}\,\|a\|_s + \|a\|_{s'}\|h\|_0 N^{\mu_0}\Big) . \tag{56}$$

Proof. We shall use several times the following bound, proved as in the first step of the proof of Lemma 3.13: if $\lambda > 0$ and $\eta > d+1$, then

$$\sum_{\sigma\in\mathcal{A}} \frac{1}{(\lambda + M_\sigma)^\eta} \leq \frac{C(\eta)}{(1+\lambda)^{\eta-(d+1)}} . \tag{57}$$

Given $h = \sum_{\alpha\in\mathcal{A}} h_\alpha$, we have

$$\mathcal{D}^{-1}\mathcal{R}h = \sum_{\sigma\in\mathcal{A}} w_\sigma \qquad \text{with} \qquad w_\sigma := (U_{\Omega_\sigma}^{\Omega_\sigma})^{-1}\Big(\sum_{\alpha\neq\sigma} U_{\Omega_\sigma}^{\Omega_\alpha} h_\alpha\Big) \in H_{\Omega_\sigma} .$$

From lemmas 3.13, 3.12 and (H2), we get, for $\nu := \delta(s - (d+1)/2)$,

$$\begin{aligned} \|w_\sigma\|_0 &\leq c\frac{M_\sigma^\tau}{\gamma}\Big(\sum_{\alpha\neq\sigma} \|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0 \|h_\alpha\|_0\Big) \\ &\leq c\frac{M_\sigma^\tau}{\gamma}\Big(\sum_{\alpha\neq\sigma} \frac{C\varepsilon\|a\|_s}{(M_\alpha + M_\sigma)^\nu}\|h_\alpha\|_0\Big) \\ &\leq C\frac{\varepsilon}{\gamma}\|a\|_s\|h\|_0 M_\sigma^\tau \Big(\sum_{\alpha\neq\sigma} \frac{1}{(M_\sigma + M_\alpha)^{2\nu}}\Big)^{1/2} \leq \frac{\varepsilon}{\gamma}\frac{C\|a\|_s}{(1+M_\sigma)^{\nu-\tau-(d+1)/2}}\|h\|_0, \end{aligned} \tag{58}$$

by the Cauchy-Schwartz inequality and (57). Hence

$$\|\mathcal{D}^{-1}\mathcal{R}h\|_0 = \Big(\sum_{\sigma\in\mathcal{A}} \|w_\sigma\|_0^2\Big)^{1/2} \le C\frac{\varepsilon}{\gamma}\Big(\sum_\sigma \frac{1}{(1+M_\sigma)^{2\nu-2\tau-(d+1)}}\Big)^{1/2}\|a\|_s\|h\|_0 \le K\frac{\varepsilon}{\gamma}\|a\|_s\|h\|_0$$

still by (57) (note that $\nu > \tau + d + 1$ by (49)). This proves (55).

In order to prove (56) we observe that:

i) if $M_\alpha \ge M_\sigma/4$ then

$$M_\sigma^{s'}\|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0\|h_\alpha\|_0 \overset{(P4)}{\le} \|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0\frac{M_\sigma^{s'}}{m_\alpha^{s'}}\|h_\alpha\|_{s'} \overset{(H1)}{\le} \|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0 2^{s'}\frac{M_\sigma^{s'}}{M_\alpha^{s'}}\|h_\alpha\|_{s'} \le \frac{C\varepsilon 8^{s'}\|a\|_s}{(M_\alpha+M_\sigma)^\nu}\|h_\alpha\|_{s'},$$

using also lemma 3.12 and (H2).

ii) if $M_\alpha < M_\sigma/4$ then, by (H1), $\mathrm{dist}(\Omega_\alpha,\Omega_\sigma) \ge M_\sigma/4$, and, by Lemma 3.12,

$$M_\sigma^{s'}\|U_{\Omega_\sigma}^{\Omega_\alpha}\|_0\|h_\alpha\|_0 \le M_\sigma^{s'}\frac{\varepsilon C(s')\|a\|_{s'}}{\mathrm{d}(\Omega_\alpha,\Omega_\sigma)^{s'-\frac{d+1}{2}}}\|h_\alpha\|_0 \le K(s')\|a\|_{s'}\varepsilon M_\sigma^{(d+1)/2}\|h_\alpha\|_0\,.$$

Therefore, from (58),

$$\begin{aligned}\|w_\sigma\|_{s'} &\overset{(P4)}{\le} M_\sigma^{s'}\|w_\sigma\|_0 \le C(s')\frac{\varepsilon}{\gamma}M_\sigma^\tau\Big(\sum_{\alpha,M_\alpha\ge M_\sigma/4}\frac{\|a\|_s\|h_\alpha\|_{s'}}{(M_\alpha+M_\sigma)^\nu} + \|a\|_{s'}M_\sigma^{(d+1)/2}\sum_{\alpha,M_\alpha<M_\sigma/4}\|h_\alpha\|_0\Big)\\ &\le C(s')\frac{\varepsilon}{\gamma}M_\sigma^\tau\Big(\frac{\|a\|_s\|h\|_{s'}}{M_\sigma^{\nu-(d+1)/2}} + \|a\|_{s'}M_\sigma^{d+1}\|h\|_0\Big), \qquad (59)\end{aligned}$$

by the Cauchy-Schwartz inequality and (57). Finally

$$\begin{aligned}\|\mathcal{D}^{-1}\mathcal{R}h\|_{s'} &= (\sum_{\sigma\in\mathcal{A}}\|w_\sigma\|_{s'}^2)^{1/2}\\ &\overset{(59)}{\le} C(s')\frac{\varepsilon}{\gamma}\Big[\Big(\sum_\sigma\frac{1}{M_\sigma^{2\nu-2\tau-(d+1)}}\Big)^{1/2}\|a\|_s\|h\|_{s'} + \Big(\sum_\sigma M_\sigma^{2\tau+2(d+1)}\Big)^{1/2}\|a\|_{s'}\|h\|_0\Big]\\ &\overset{(57)}{\le} K(s')\frac{\varepsilon}{\gamma}\Big(\|h\|_{s'}\|a\|_s + \|a\|_{s'}N^{\tau+3(d+1)/2}\|h\|_0\Big)\end{aligned}$$

proving (56). ■

By (55), for $\varepsilon\gamma^{-1}\|a\|_s$ small, the operator U is invertible, and, using (56), we prove the following interpolation type inequality for U^{-1}.

Lemma 3.16 *$\forall s' \ge s(d,\tau)$, there is $c(s') > 0$ such that, if $\varepsilon\gamma^{-1}(\|a\|_s + 1) \le c(s')$, then*

$$\|U^{-1}h\|_{s'} \le K(s')\gamma^{-1}N^\mu(\|h\|_{s'} + \|a\|_{s'}\|h\|_0)\,, \quad \forall h \in H_S\,,$$

with $\mu := 2\tau + 3(d+1)/2$.

Proof. If $\varepsilon\|a\|_s\gamma^{-1}$ is small enough then, by (55), $\|D^{-1}\mathcal{R}\|_0 \le 1/2$ and $I + D^{-1}\mathcal{R}$ is invertible, $\|(I + D^{-1}\mathcal{R})^{-1}\|_0 \le 2$. Then, arguing like in the proof of Lemma 3.8, we derive from (56) that for $\varepsilon(\|a\|_s + 1)\gamma^{-1} \le c(s')$ small enough,

$$\|(I + \mathcal{D}^{-1}\mathcal{R})^{-1}h\|_{s'} \le 2\|h\|_{s'} + \|a\|_{s'}\|h\|_0 N^{\mu_0}\,.$$

Using lemma 3.14 the thesis follows. ■

Proof of Proposition 3.1 concluded. Lemma 3.16, (46), (45) and lemma 3.8 yield (37). We use also that $\|a\|_{s'} = \|(\partial_u F)(t,x,u)\|_{s'} \le C(s')\|u\|_{s'}$, by property $(P2)$ applied to $\partial_u F$. ■

4 Separation properties of the singular sites

In this section we verify, assuming only the standard Diophantine condition (7) and the nonresonance condition (2), that, for $\rho < \rho(\gamma)$, there is a partition of the singular sites like in (35) verifying (H1)-(H2) with δ depending only on d. The proof follows essentially the scheme of [6] except in lemmas 4.1 and 4.4.

Consider the bilinear symmetric form $\varphi_\omega : \mathbf{R}^{d+1} \times \mathbf{R}^{d+1} \to \mathbf{R}$ defined by

$$\varphi_\omega(x, x') := j \cdot j' - \omega^2 l\, l' , \qquad \forall\, x = (l, j) ,\ x' = (l', j') \in \mathbf{R} \times \mathbf{R}^d$$

and the corresponding quadratic form

$$Q_\omega(x) = \varphi_\omega(x, x) := |j|^2 - \omega^2 l^2 .$$

A vector $x = (l, j) \in \mathbf{Z} \times \mathbf{Z}^d$ is said "weakly singular" if $|Q_\omega(x)| \le 1 + |m|$.

Definition 4.1 *A sequence $x_0, \ldots, x_K \in \mathbf{Z}^{d+1}$ of distinct, weakly singular, integer vectors satisfying, for some $B \ge 2$, $|x_{k+1} - x_k| \le B$, $\forall k = 0, \ldots, K-1$, is called a B-chain of length K.*

Theorem 4.1 *If ω^2 satisfies (7), then any B-chain has length $K \le B^C/\gamma^r$ for some $C := C(d) > 0$ and $r := r(d) > 0$.*

Remark 4.1 *Theorem 4.1 has been proved in [6], see lemma 2.10, assuming the stronger Diophantine condition $|\sum_{j=0}^{10d} a_j \omega^{2j}| \ge (\sum_j |a_j|)^{-C}$, $\forall (a_j) \in \mathbf{Z}^{10d+1} \backslash \{0\}$ and in [10], lemma 8.7, assuming $|\sum_{j=0}^{d+1} a_j \omega^{2j}| \ge (\sum_j |a_j|)^{-C}$, $\forall (a_j) \in \mathbf{Z}^{d+1} \backslash \{0\}$.*

The proof of Theorem 4.1 is split in several lemmas.

Given integer vectors $f_i \in \mathbf{Z}^{d+1}$, $i = 1, \ldots, n$, $1 \le n \le d+1$, linearly independent on $\mathbf{R}$, we consider the subspace $F := \mathrm{Span}_{\mathbf{R}}\{f_1, \ldots, f_n\}$ of $\mathbf{R}^{d+1}$ and the restriction $\varphi_{\omega|F}$ of the bilinear form φ_ω to F, which is represented by the symmetric matrix

$$A_\omega := \{\varphi_\omega(f_i, f_{i'})\}_{i,i'=1}^n \in \mathrm{Mat}_n(\mathbf{R}) .$$

Introducing the symmetric bilinear forms

$$\mathcal{R}(x, x') := j \cdot j' \quad \text{and} \qquad \mathcal{S}(x, x') := l\, l',$$

we write

$$\varphi_\omega = \mathcal{R} - \omega^2 \mathcal{S} \ \text{ and } \ A_\omega = R - \omega^2 S$$

where $R := \{\mathcal{R}(f_i, f_{i'})\}_{i,i'=1}^n = (R_1, \ldots, R_n)$, $S := \{\mathcal{S}(f_i, f_{i'})\}_{i,i'=1}^n = (S_1, \ldots, S_n)$ are the matrices that represent respectively $\mathcal{R}_{|F}$ and $\mathcal{S}_{|F}$ in the basis $\{f_1, \ldots, f_n\}$. Note that the matrices R, S have integer coefficients. Here $R_i, S_i \in \mathbf{Z}^n$, $i = 1, \ldots, n$ denote the column vectors respectively of R and S. The following lemma is the main difference with respect to [6].

Lemma 4.1 *Assume that ω^2 satisfies (7). Then A_ω is invertible and*

$$\|A_\omega^{-1}\| \le \frac{c(n)}{\gamma} \Big(\max_{i=1,\ldots,n} |f_i| \Big)^{5n-2} . \tag{60}$$

Proof. The matrix S has rank at most 1 because it represents the restriction to F of a bilinear form of rank 1. Since any two columns of S are colinear, the development in ω^2 of $\det A_\omega$ reduces to

$$\det A_\omega = \det(R_1 - \omega^2 S_1, \ldots, R_n - \omega^2 S_n) = \det(R_1, \ldots, R_n) - \omega^2 \sum_{i=1}^n \det(R_1, \ldots, S_i, \ldots, R_n). \tag{61}$$

Therefore $\det A_\omega = P(\omega^2)$ is a polynomial in ω^2 of degree at most 1, with integer coefficients since $\det R$, $\det(R_1, \ldots, S_i, \ldots, R_n) \in \mathbf{Z}$. Furthermore $P(\cdot)$ is not identically zero because $P(-1) = \det(R + S)$ is positive (the matrix $R + S$ being positive definite).

By (61), if $\det R = 0$ then $|\det A_\omega| \geq \omega^2$, and, if $\det R \neq 0$, since ω^2 satisfies (7), $|\det A_\omega| \geq \gamma |\det R|^{-3/2}$. Since $R + S = {}^t\mathcal{F}\,\mathcal{F}$ with $\mathcal{F} = (f_1, \ldots, f_n)$, we have $0 \leq \det R \leq \det(R+S) = (\det \mathcal{F})^2 \leq |f_1|^2 \ldots |f_n|^2 \leq M^{2n}$, where $M := \max_{i=1,\ldots,n} |f_i|$. Hence

$$|\det A_\omega| \geq \frac{\gamma}{M^{3n}} . \tag{62}$$

By (7), $\omega^2 \geq \gamma$, and (62) holds in both cases. Applying the Cramer rule

$$|(A_\omega^{-1})_{i,i'}| \leq \frac{c(n)}{|\det A_\omega|}(M^2)^{(n-1)} \stackrel{(62)}{\leq} \frac{c'(n)}{\gamma} M^{3n+2(n-1)}$$

whence (60) follows. ■

The remainder of the proof follows [6] (although some details are different). We report it for completeness.

By lemma 4.1 the bilinear form $\varphi_{\omega|F}$ is non-degenerate and we decompose

$$\mathbf{R}^{d+1} = F \oplus F^{\perp \varphi_\omega} \qquad \text{where} \qquad F^{\perp \varphi_\omega} := \Big\{ x \in \mathbf{R}^{d+1} \mid \varphi_\omega(x, f) = 0 , \quad \forall f \in F \Big\} .$$

We denote by $P_F : \mathbf{R}^{d+1} \to F$ the corresponding projector onto F.

Lemma 4.2 *Let $x_0, \ldots, x_L$ be a B-chain. Define $G := \mathrm{Span}_{\mathbf{R}}\{x_i - x_k \mid i, k = 0, \ldots, L\} \subset \mathbf{R}^{d+1}$. Then, calling $n := \dim G$,*

$$|P_G(x_j)| \leq \frac{c(n)}{\gamma}(LB)^{5n+1} , \qquad \forall j = 0, \ldots, L . \tag{63}$$

Proof. For all $i \neq j$, by bilinearity,

$$Q_\omega(x_i) = Q_\omega(x_j + (x_i - x_j)) = Q_\omega(x_j) + 2\varphi_\omega(x_j, x_i - x_j) + Q_\omega(x_i - x_j) . \tag{64}$$

Since $|x_i - x_j| \leq |i - j| B$ and $|Q_\omega(x_i)|$, $|Q_\omega(x_j)| \leq 1 + |m|$, we get

$$|2\varphi_\omega(x_j, x_i - x_j)| \stackrel{(64)}{\leq} 2 + 2|m| + C|x_i - x_j|^2 \leq 2 + 2|m| + C|i - j|^2 B^2 \leq C' L^2 B^2 . \tag{65}$$

According to the decomposition $\mathbf{R}^{d+1} = G \oplus G^{\perp \varphi_\omega}$ we write $x_j = y_j + z_j$ with $y_j := P_G(x_j) \in G$ and $z_j \in G^{\perp \varphi_\omega}$.

Fixed any $j \in [0, L]$ we have $G = \mathrm{Span}_{\mathbf{R}}\{x_i - x_j \ ; \ i = 0, \ldots, L\}$. Let us extract from $\{x_i - x_j, i = 0, \ldots, L\}$, a basis $\{f_1, \ldots, f_n\}$ of G. We develop $y_j = \sum_{l=1}^n a_{l,j} f_l$ whence

$$A_\omega a = b \quad \text{where} \quad a := \begin{pmatrix} a_{1,j} \\ \ldots \\ a_{n,j} \end{pmatrix} , \quad b := \begin{pmatrix} \varphi_\omega(f_1, x_j) \\ \ldots \\ \varphi_\omega(f_n, x_j) \end{pmatrix} , \tag{66}$$

using that $\varphi_\omega(f_m, y_j) = \varphi_\omega(f_m, x_j)$ because $z_j \in G^{\perp \varphi_\omega}$. By (65), $|b| \leq C L^2 B^2$ and, by lemma 4.1,

$$|a| \leq \|A_\omega^{-1}\| |b| \leq \frac{c(n)}{\gamma} \Big(\max_{i=1,\ldots,n} |f_i| \Big)^{5n-2} L^2 B^2 \leq \frac{c(n)}{\gamma}(LB)^{5n}$$

since $|f_i| \leq LB$. Finally, $|y_j| \leq \sum_{l=1}^n |a_{l,j}| \, |f_l| \leq (c(n)/\gamma)(LB)^{5n} LB$, which gives (63). ■

Lemma 4.3 *Let $x_0, \ldots, x_K$ be a B-chain. Let $1 \leq L \leq K$ be such that*

$$\forall j = 0, \ldots, K , \quad \mathrm{Span}_{\mathbf{R}}\Big\{ x_i - x_j \mid |i - j| \leq L \Big\} = \mathrm{Span}_{\mathbf{R}}\Big\{ x_i - x_l \mid i, l = 0, \ldots, K \Big\} =: F . \tag{67}$$

Then, for $\rho_n := 5n + 1$, $n := \dim F$,

- *a)* $|P_F(x_j)| \le \frac{c(n)}{\gamma}(LB)^{\rho_n}$, $\forall j = 0, \dots, K$;
- *b)* $|x_{j_2} - x_{j_1}| \le \frac{c(n)}{\gamma}(LB)^{\rho_n}$, $\forall j_1, j_2 \in \{0, \dots, K\}$;
- *c)* $K \le \frac{c(n)}{\gamma^n}(LB)^{\rho_n n}$.

PROOF. For every fixed $j \in \{0, \dots, K\}$ apply lemma 4.2 to the B-chain $\{x_i\}$ with $|i-j| \le L$, $i \in \{0, \dots, K\}$. By (67) we have $G := \mathrm{Span}_{\mathbf{R}}\{x_i - x_k \mid |i-j|, |k-j| \le L\Big\} = F$, and therefore, by (63) we deduce $a)$.

Now, by the definition of F, $\forall j_1, j_2 \in \{0, \dots, K\}$, we have $x_{j_2} - x_{j_1} \in F$ and therefore by $a)$,

$$|x_{j_2} - x_{j_1}| = |P_F(x_{j_2} - x_{j_1})| \le |P_F(x_{j_2})| + |P_F(x_{j_1})| \le \frac{c(n)}{\gamma}(LB)^{5n+1},$$

which is $b)$.

Finally, $\forall j = 1, \dots, K$, by $b)$, $|x_j - x_0| \le (c(n)/\gamma)(LB)^{\rho_n}$, whence x_j belongs to the n-dimensional ball centered at x_0 with radius $(C(n)/\gamma)(LB)^{\rho_n}$. The number of integer vectors inside such a ball is less or equal to $(C'(n)/\gamma^n)(LB)^{\rho_n n}$ and we deduce $c)$. ■

PROOF OF THEOREM 4.1 CONCLUDED. Define $F := \mathrm{Span}_{\mathbf{R}}\{x_i - x_l \mid i, l = 0, \dots, K\} \subset \mathbf{R}^{d+1}$.

If $\mathrm{dim}F = 1$ then the B-chain $x_0, \dots, x_K$ satisfies (67) with $L = 1$ because all the x_i are distinct. By $c)$ we get $K \le (c/\gamma)B^6$. Next, suppose by induction that Theorem 4.1 holds if $\mathrm{dim}F \le n$, $n \le d$. We want to prove Theorem 4.1 when $\mathrm{dim}F = n+1$.

Fix $L = [K^{1/\alpha}]$ with $\alpha := 2\rho_{n+1}(n+1)$. If the B-chain $x_0, \dots, x_K$ satisfies (67), then, by $c)$,

$$K \le \frac{c(n+1)}{\gamma^{n+1}}(LB)^{\rho_{n+1}(n+1)} \le \frac{c(n+1)}{\gamma^{n+1}}\sqrt{K}B^{\rho_{n+1}(n+1)} \quad \Longrightarrow \quad K \le \frac{c'(n+1)}{\gamma^{2(n+1)}}B^{2\rho_{n+1}(n+1)} \le \frac{B^{C(n)}}{\gamma^{r(n)}}.$$

Otherwise, there exists $j \in \{0, \dots, K\}$ such that $\dim \mathrm{Span}_{\mathbf{R}}\{x_i - x_j \mid |i-j| \le L\} \le n$. Consider the B-chain $\{x_i\}$ with $|i-j| \le L$ whose length is at most $2L$. By the inductive assumption, $2L \le B^{C(n-1)}/\gamma^{r(n-1)}$, whence $K \le (L+1)^\alpha \le cB^{\alpha C(n-1)}/\gamma^{\alpha r(n-1)} \le B^{C(n)}/\gamma^{r(n)}$. In both cases $K \le B^C/\gamma^r$ for some $C := C(n)$, $r = r(n)$. ■

Finally we show how to get the required decomposition of the singular sites in clusters.

Lemma 4.4 *Assume (2) and (7). $\forall \gamma \in (0,1)$, if $\rho \le \rho(\gamma) := \gamma^{3r(d)+1}/8$ there exists a decomposition of the singular sites*

$$S := \Big\{(l,j) \in \Omega_N \ : \ |\omega^2 l^2 - \lambda_j| < \rho\Big\} = \bigcup_{\alpha \in \mathcal{A}} \Omega_\alpha \tag{68}$$

in clusters, like in (35), satisfying properties (H1)-(H2) with $\delta := 1/2(C(d)+1)$ (the constants $C(d), r(d)$ are those of Theorem 4.1).

PROOF. We introduce the following equivalence relation.

Definition 4.2 *Given $\delta > 0$, two integer vectors $x, y \in S$ are said equivalent if there exist $x_l \in S$, $l = 0, 1, \dots n$, with $x_0 = x$, $x_n = y$ and $|x_{l+1} - x_l| \le (|x_l| + |x_{l+1}|)^\delta$, $\forall l$.*

Chosen $\delta := \delta(d) := 1/2(C(d)+1)$, where $C(d)$ is the constant which appears in Theorem 4.1, we get a decomposition of S in disjoint equivalence classes Ω_α as in (68). To verify that each Ω_α is dyadic, consider $z_\alpha \in \Omega_\alpha$ such that $|z_\alpha| = \max_{z \in \Omega_\alpha} |z| = M_\alpha$. Each $x \in \Omega_\alpha$ is connected to z_α by a B-chain with $B = (2M_\alpha)^\delta$. Hence, by Theorem 4.1, there are no more than B^C/γ^r elements in this chain. Therefore $\forall x \in \Omega_\alpha$,

$$|x| \ge |z_\alpha| - \frac{B^C B}{\gamma^r} = M_\alpha - \frac{(2M_\alpha)^{\delta(C+1)}}{\gamma^r} = M_\alpha - \frac{\sqrt{2M_\alpha}}{\gamma^r} \ge \frac{M_\alpha}{2},$$

provided that $M_\alpha \geq 8/\gamma^{2r}$. As a consequence, Ω_α satisfies the dyadic property (H1) whenever $M_\alpha \geq 8/\gamma^{2r}$. Now, if $|x| = |(l,j)| \leq 8/\gamma^{2r}$ then by (2)

$$|\omega^2 l^2 - \lambda_j| \geq \frac{3\gamma}{(8\gamma^{-2r})^{3/2}} \geq \frac{\gamma^{3r+1}}{8}.$$

As a result, if $\rho \leq \rho(\gamma) := \gamma^{3r+1}/8$ then there is no singular site in the open ball of center 0 and radius $8/\gamma^{2r}$, and all the Ω_α satisfy $M_\alpha \geq 8/\gamma^{2r}$ and are dyadic.

To prove (H2) consider $x_\alpha \in \Omega_\alpha$, $x_\sigma \in \Omega_\sigma$ such that $\mathrm{d}(\Omega_\alpha, \Omega_\sigma) = |x_\alpha - x_\sigma|$. Since $x_\sigma \notin \Omega_\alpha$

$$|x_\alpha - x_\sigma| > (|x_\alpha| + |x_\sigma|)^\delta \geq (m_\alpha + m_\sigma)^\delta \geq \frac{1}{2}(M_\alpha + M_\sigma)^\delta$$

by (H1), that is (H2). ■

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